Homework #6 for MATH 8301: Manifolds and Topology

October 17, 2017

Due Date: Monday 23 October in class.

1. Let $\{G_\alpha\}_{\alpha \in A}$ be a set of groups indexed by A, and write

$$G := \bigstar_{\alpha \in A} G_\alpha$$

for the free product of the G_α. Let H be any group, and write $\mathrm{Hom}(G, H)$ for the *set* of group homomorphisms from G to H. Prove that there is a bijection

$$\mathrm{Hom}(G, H) \cong \prod_{\alpha \in A} \mathrm{Hom}(G_\alpha, H),$$

where the target is the product of the sets $\mathrm{Hom}(G_\alpha, H)$.

2. Let X be a set; a *binary operation* on X is a map $\mu : X \times X \to X$. Assume that X has *two* binary operations; we'll write them as

$$(x, y) \mapsto x \star y \quad \text{and} \quad (x, y) \mapsto x \cdot y.$$

Assume that both $\star$ and $\cdot$ are unital: there are elements $1_\star$ and $1_.$ with

$$x \star 1_\star = x = 1_\star \star x \quad \text{and} \quad x \cdot 1_. = x = 1_. \cdot x.$$

Also assume that $\star$ and $\cdot$ interact via:

$$(x \cdot y) \star (w \cdot z) = (x \star w) \cdot (y \star z) \tag{1}$$

Hints: For all the following, do a lot of multiplying by 1, and invoking Equation (1).

(a) Prove that $1_. = 1_\star$.

(b) Prove that $a \cdot b = b \star a$ and that $a \cdot b = a \star b$. That is: $\star$ and $\cdot$ are commutative, and are equal.

(c) Prove that $\star$ (and hence $\cdot$) is associative.

3. Let G be a topological space with a continuous binary operation $\mu : G \times G \to G$ and an element $e \in G$ with the property[1] that $\mu(g, e) = g = \mu(e, g)$. Let γ and ρ be loops in G based at e, and define

$$(\gamma \cdot \rho)(t) = \mu(\gamma(t), \rho(t)).$$

   (a) Define a binary operation $\cdot$ on $\pi_1(G, e)$ as $[\gamma] \cdot [\rho] = [\gamma \cdot \rho]$. Verify that this is well-defined.

   (b) Let $1. \in \pi_1(G, e)$ be the homotopy class of the constant loop at e. Show that $[\gamma] \cdot 1. = [\gamma] = 1. \cdot [\gamma]$.

   (c) Let $\star$ be the binary operation on $\pi_1(G, e)$ coming from concatenation of loops. Show that $\star$ and $\cdot$ satisfy Equation (1).

   (d) Prove that $\pi_1(G, e)$ is an abelian group (using the usual multiplication of concatenation of loops).

4. Let X be the space

$$X = \{x \in \mathbb{R}^3 \mid 1 \leq |x| \leq 2\} \subseteq \mathbb{R}^3.$$

   X has two boundary components, S_1 and S_2, consisting of those elements of norm 1 and 2, respectively. Generate an equivalence relation $\sim$ on X by setting $x \sim y$ if $x \in S_1$, $y \in S_2$, and $y = 2x_1$. Compute $\pi_1(X/\sim, x_0)$ for any point $x_0 \in X/\sim$.

[1] G could be, for instance, a topological group.