Homework #4 for MATH 8301: Manifolds and Topology

September 26, 2017

Due Date: Monday 2 October in class.

1. For a 2-dimensional simplicial complex $(V, \mathcal{F})$ with v vertices, e edges, and f triangles, the *Euler characteristic* χ is defined to be

$$\chi = v - e + f.$$

   (a) If the geometric realization of $(V, \mathcal{F})$ is a compact *surface*, find a relation between e and f.

   (b) Using the previous part, give formulas for e and f as functions of χ and v, and show that they are nondecreasing in v .

   (c) Using the formulas from the previous problem (possibly repeatedly), show that any triangulation of a compact surface of Euler characteristic 0 requires at least 7 vertices. **Hints:** Is there an extremely naive lower bound on the number of vertices of a 2 dimensional complex? Is there an upper bound on the number of edges as a function of the number of vertices?

2. A region $P \subseteq \mathbb{R}^2$ in the plane is said to be *star-shaped* with respect to a point $p \in P$ if for every $q \in p$, the straight line $\overline{pq}$ from p to q is contained in P.

   (a) Show that if P is star-shaped, then it is contractible.

   (b) If P is a polygon which is star-shaped with respect to a point p in the interior of P, define function

$$f : P \setminus \{p\} \to S^1 \text{ via } f(q) = \frac{q - p}{|q - p|}$$

   and show that f is a homotopy equivalence.

   (c) Prove that if $p \in T^2$, there is a homotopy equivalence

$$T^2 \setminus p \simeq S^1 \vee S^1$$

   from the torus punctured at p to the wedge of two circles (here, the *wedge* of spaces X and Y with respect to two points $x \in X$ and $y \in Y$ is $X \vee Y := X \sqcup Y / \sim$, where $\sim$ is the equivalence relation which *only* identifies $x \sim y$.