Homework #11 for MATH 8301: Manifolds and Topology

November 29, 2017

Due Date: Wednesday 6 December in class.

1. Recall from Hatcher (Prop 1.40 and just before it) that an action of a group G on a space $\overline{X}$ is a *covering space action* if each $x \in \overline{X}$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. Then letting $X = \overline{X}/G$ be the quotient space under this action, the quotient map $\overline{X} \to X$ is a normal covering space, and the group of deck transformations $\mathrm{Aut}(\overline{X}/X) \cong G$ is isomorphic to G.

   Each subgroup $H \subseteq G$ determines a composition of covering spaces $\overline{X} \to \overline{X}/H \to \overline{X}/G$. Show:

   (a) Every path-connected covering space between $\overline{X}$ and $\overline{X}/G$ is isomorphic to X/H for some subgroup $H \subseteq G$.

   (b) Two such covering spaces $\overline{X}/H_1$ and $\overline{X}/H_2$ of $\overline{X}/G$ are isomorphic iff H_1 and H_2 are conjugate subgroups of G.

   (c) The covering space $\overline{X}/H \to \overline{X}/G$ is normal iff H is a normal subgroup of G, in which case the group of deck transformations of this cover is G/H.

2. Given a group G and a normal subgroup N, show that there exists a normal covering space $\overline{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\overline{X}) \cong N$, and deck transformation group $\mathrm{Aut}(\overline{X}/X) \cong G/N$. You are welcome to assume that G is finitely presented if that's helpful.

3. Show that chain homotopy of chain maps is an equivalence relation.

4. Show that if X retracts onto a subspace $A \subseteq X$, then the map $H_*(A) \to H_*(X)$ is injective.

5. Let A be any finitely generated abelian group. Construct a chain complex C_* with the property that $H_0(C_*) \cong A$, but $H_j(C_*) = 0$ for all $j \neq 0$.