Homework #10 for MATH 8301: Manifolds and Topology

November 15, 2017

Due Date: Wednesday 22 November in class.

1. Let X be the union of the unit sphere $S^2 \subseteq \mathbb{R}^3$ and one of its diameters (say $[-1,1] \times \{(0,0)\}$, for definiteness).
   (a) Compute $\pi_1(X)$.
   (b) Construct (geometrically) a simply connected covering space of X, $\tilde{X}$.
   (c) Enumerate the set of subgroups of $\pi_1(X)$.
   (d) Compute the set of isomorphism classes of covers of X, and describe each covering space as a topological space (pictures or descriptive language is sufficient, and in fact, more helpful than formulas).
2. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$.
   (a) Show that $\pi_1(X) = \mathbb{Z}/2 * \mathbb{Z}/2$.
   (b) Construct (geometrically) a simply connected covering space of X, $\tilde{X}$.
   (c) Form the semidirect product $G := \mathbb{Z}/2 \ltimes \mathbb{Z}$ for the action of $\mathbb{Z}/2$ on $\mathbb{Z}$ by negations: elements are pairs (a, x), for $a \in \mathbb{Z}/2$ and $x \in \mathbb{Z}$, with multiplication
   $$(a, x) * (b, y) = (a + b, (-1)^b * x + y)$$
   (here I'm using additive notation in both $\mathbb{Z}/2$ and $\mathbb{Z}$). Construct an isomorphism between G and $\mathbb{Z}/2 * \mathbb{Z}/2$.
   (d) Enumerate the set of subgroups of G.
   (e) As in the previous problem, compute the set of *based* isomorphism classes of covers of X, and describe each cover as a space.