Homework #8 for MATH 5345H: Introduction to Topology

November 11, 2019

Due Date: Monday 18 November in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills.

This week we will focus on finding and avoiding errors in our proof-writing. You'll be asked to criticize and improve an argument in the first problem. Please do keep this focus in mind as you work on this the other problems this week – are your proofs as clear and compelling as they could be? What can be improved?

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1. Consider the following proposition and its proof:

   Proposition 1. *Let $f : X \to Y$ be a homeomorphism. Then X is compact if and only if Y is.*

   Proof. Since f is a homeomorphism, it sets up a bijection between the open sets of X and the open sets of Y. Therefore an open covering of X admits a finite subcovering if and only if the corresponding covering of Y admits a finite subcovering. The result follows. $\square$

   Identify some of the flaws in this argument, and rewrite it so that it is more convincing. Alternatively to rewriting the proof (or additionally) find a different, more compelling proof.

2. Show that any set equipped with the finite complement topology is compact.

3. Let C be the Cantor set from the previous homework. Show that C is compact.

4. Let X and Y be spaces, and let $x_0 \in X$, $y_0 \in Y$ be closed points. Let $X \amalg Y$ be the disjoint union of X and Y (so that an open set of $X \amalg Y$ is the union of an open set of X and an open set of Y) and define the *wedge sum* $X \vee Y$ by

$$X \vee Y = (X \amalg Y)/(x_0 = y_0).$$

The notation means the quotient of $X \amalg Y$ by the equivalence relation $\sim$ with $x_0 \sim y_0$ and no other nontrivial equivalence classes - so $X \vee Y$ is obtained by joining X and Y at a point. The space $X \vee Y$ depends on the choices of x_0 and y_0, but we suppress this from the notation.

If X and Y are connected, show that $X \vee Y$ is connected. If X and Y are compact, show that $X \vee Y$ is compact.

5. Give $\mathbb{Q}$ the subspace topology of $\mathbb{R}$. Is it locally compact? Prove your answer.