Homework #6 for MATH 5345H: Introduction to Topology

October 13, 2019

Due Date: Friday 18 October in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills.

One of the harder parts of work as a research mathematician is trying to solve problems where the answer is unknown *to anyone*. Proving something is true when you know *what* to prove is immensely easier than trying to figure out what is true and then to go prove it. The open-endedness of such a problem can be confounding.

In problem 3 below, we've created a scenario like this, where you are asked to find a criterion which establishes a desired result, and then to go verify that it does what you want it to do. You may find the unstructured nature of the problem overwhelming, or you may see a solution very quickly. Regardless, it will usually take some careful thought to turn your intuition into written form that is digestible by others. As you grapple with this problem, consider: what is the best way to present your ideas for an audience which has not thought about this question before?

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1. Endow $\mathbb{R}$ with its usual topology.

   (a) Show that the sum and product functions

   $$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ f(x, y) = x + y$$

   and

   $$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ g(x, y) = xy$$

   are continuous.

   (b) Let $\overline{\mathbb{R}}$ be the topological space whose underlying set is $\mathbb{R}$ and whose open sets are $\emptyset$, $\mathbb{R}$ and the rays

   $$(a, \infty) = \{r \in \mathbb{R} \,|\, r > a\}$$

for each $a \in \mathbb{R}$. Are the sum and product functions

$$\overline{f} : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}},\ \overline{f}(x, y) = x + y$$

and

$$g : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}},\ \overline{g}(x, y) = xy$$

continuous? Prove or disprove.

In this question it's OK to be informal about whether a subset of $\mathbb{R}^2$ is open.

2. A subset $A \subseteq X$ is said to be *dense* if $\overline{A} = X$. If $A \subseteq X$ and $B \subseteq Y$ are both dense, show that $A \times B$ is dense in $X \times Y$ (in the product topology).

3. Let X be a set.

   (a) Consider the finite complement topology on X. Give a criterion on the cardinality of X which characterizes precisely when this topology is Hausdorff. That is, find a statement **(S)** about the cardinality of X with the property that **(S)** is true if and only if X (with the finite complement topology) is Hausdorff.

   (b) Consider the following statement **(O)**:

   > **(O)** There is a total order $<$ on X with the property that the order topology on X is equal to the finite complement topology on X.

   Use the results last problem of HW 5 to show that your statement **(S)** implies the statement **(O)**.

   (c) Is the converse true? That is, does statement **(O)** imply statement **(S)**? If this is true, prove it; if false, give a counterexample.