Homework #5 for MATH 5345H: Introduction to Topology

October 13, 2019

Due Date: Monday 14 October in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills.

It's often the case that a correct proof can be improved with a picture of some sort. That is: the guts of proving a mathematical fact usually lies in set-theoretic or logical arguments. But these arguments can sometimes be hard to follow without a visual guide of some sort. This is particularly true in topology and geometry. In this week's homework, we'll ask you to illustrate two problems (2 and 3) with a picture that illuminates the idea of the proof. Your pictures can be hand-drawn or electronically created.

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1. A continuous map $f : X \to Y$ is called an *open map* if for every open set $U \subseteq X$, its image $f(U) \subseteq Y$ is open. Similarly, f is called a *closed map* if for every closed set $Z \subseteq X$, its image $f(Z) \subseteq Y$ is closed.

   Suppose K is a subset of X, equipped with its subspace topology, and let $i : K \to X$ be the inclusion of K into X. Show that i is an open map if and only if K is an open subset of X, and that i is a closed map if and only if K is a closed subset of X.

2. Let $X = \mathbb{Z}_{\geq 0} \times [0, 1)$. Both $\mathbb{Z}_{\geq 0}$ and $[0, 1)$ are ordered sets, coming from the order on $\mathbb{R}$, by restriction to these subsets. Equip X with the dictionary order, and the order topology coming from this order. Let $Y = [0, \infty) \subseteq \mathbb{R}$ be the closed ray in $\mathbb{R}$, equipped with the subspace topology. This question aims to show that X and Y are homeomorphic.

   (a) Define $f : X \to Y$ by the formula $f(n, t) = n + t$. Define $g : Y \to X$ by $g(x) = (\lfloor x \rfloor, x - \lfloor x \rfloor)$, where $\lfloor x \rfloor$ is the greatest integer less than x. Verify that these are mutually inverse bijections.

   (b) Show that f and g are continuous, and conclude that X and Y are homeomorphic.

(c) Please illustrate your argument with a picture of some sort which helps explain the proof.

3. Let X be a topological space, and let $\Delta \subseteq X \times X$ be the *diagonal*:

$$\Delta := \{(x, x) \mid x \in X\} \subseteq X \times X$$

Show that X is Hausdorff if and only if Δ is a closed subset of $X \times X$.

Please illustrate your argument with a picture of some sort which helps explain the proof. This may be harder than in the previous problem; your picture will by necessity be somewhat heuristic.

4. Show that every order topology is Hausdorff.