Homework #3 for MATH 5345H: Introduction to Topology

September 23, 2019

Due Date: Friday 27 September in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills. This week, I would like you to focus on the structure of sentences within your proof. Strong proofs are written in paragraphs that comprise complete sentences. Here are some qualities that sentences in mathematical writing should have. Sentences should:

1. start with a capital letter (and in particular, a word and not a variable),
2. end with with a period,
3. contain a subject and a verb, and
4. comprise words instead of symbols.

On the last point: certainly symbols do belong in mathematical writing:

> "Let $f : X \to Y$ be a continuous map."

is a perfectly valid mathematical sentence. In contrast,

> "$\forall x, \exists y$ s.t. $y > x$"

could certainly be improved to "For all x, there exists a y such that $y > x$" if it occurs in the midst of an argument. On the other hand, you're perfectly welcome to use such shorthand in defining sets, for instance. For example,

$$S := \{f : \mathbb{R} \to \mathbb{R} \mid \exists N \, s.t. \, \forall x, f(x) < N\}$$

describes the set of bounded-above functions on $\mathbb{R}$. Getting this right can be difficult, but the theme is certainly: if you're writing an explanation, please make it a sentence.

Here are some examples of statements from proofs written in 5XXX-level math courses that **do not** constitute proper sentences. Can you identify why each of these statements do not belong in a mathematical proof?

- 109 is a factor! $109 - 1 = 108 = 2^2 3^2$ is 3-smooth.
- $\implies y$ is a nonsquare
- group of T persons $a_1, \ldots, a_T =$ heights of T persons
- $Y = \#$ nondefective “”
- goal: find $P(X_2 = 1 | X_1 = 1)$
- $\therefore$ independent

Please keep this in mind as you approach this week's homework:

1. Let X be a set, and let T be a topology on X in which the singleton $\{x\} \in T$ for each $x \in X$. Show that T is the discrete topology on X.

2. Let $\mathbb{R}[x_1, \ldots, x_n]$ denote the set of polynomials in n variables $x_1, \ldots, x_n$ whose coefficients lie in $\mathbb{R}$. So, for instance, $x_1 - 3x_2^2 + \sqrt{2}x_7^4 \in \mathbb{R}[x_1, \ldots, x_9]$, but neither $\frac{x_1}{x_2}$ nor ix_5^3 is an element of this set of polynomials.

   For a subset $S \subseteq \mathbb{R}[x_1, \ldots, x_n]$, write $V(S) \subseteq \mathbb{R}^n$ to be the set

$$\begin{aligned} V(S) &= \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = 0, \forall f \in S\} \\ &= \bigcap_{f \in S} \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = 0\}. \end{aligned}$$

   Let $U(S) = \mathbb{R}^n \setminus V(S)$.

   We asserted in class that the collection $T_{Zar} = \{U(S),\ S \subseteq \mathbb{R}[x_1, \ldots, x_n]\}$ forms a topology on $\mathbb{R}^n$, called the *Zariski topology*, but some details were definitely missing. We'll fill in those details here:

   (a) For any two sets $S, T \subseteq \mathbb{R}[x_1, \ldots, x_n]$, define

$$ST := \{f \cdot g \mid f \in S, g \in T\}.$$

   Show that $V(ST) = V(S) \cup V(T)$.

   (b) Show that T_{Zar} is a topology on $\mathbb{R}^n$. You should feel free to use the following facts (which we established in class) without proof:

   - For any real number r (such as $r = 0$ or $r = 1$), write r for the constant polynomial r. Then

$$V(\{0\}) = \mathbb{R}^n \text{ and } V(\{1\}) = \emptyset.$$

   - For any indexing set J,

$$V(\bigcup_{j \in J} S_j) = \bigcap_{j \in J} V(S_j)$$

(c) Fix $n = 1$, and show that for any set $S \subseteq \mathbb{R}[x_1]$, $V(S)$ is finite. Conversely, let $F \subset \mathbb{R}$ be any finite set. Find a set $T \subseteq \mathbb{R}[x_1]$ with $V(T) = F$.

(d) Show that the Zariski topology on $\mathbb{R}^1$ is equal to the finite complement topology.

3. Show that if $\mathcal{B}$ is a basis for a topology on X, then the topology generated by $\mathcal{B}$ equals the intersection of all topologies on X that contain $\mathcal{B}$. Is the same true for a subbasis?