# Homework #5 for MATH 5345H: Introduction to Topology

November 3, 2018

**Due Date:** Friday 9 November in class.

**Focus on writing:** Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills.

Last week, I encouraged you to focus on sentence structure, including when to use words instead of symbols. One way to ensure this might be to replace all symbols with words, however, this can lead to proofs that are lengthy and difficult to read. Better writing does not mean more words! This week, I would like you to focus on the concision of your sentences. Below is an unnecessarily verbose proof of one of the propositions we proved in class Friday. Can you identify how to make this proof more concise without sacrificing sentence structure?

**Proposition 1.** *If $X$ is a space, $A \subseteq X$ is a connected subspace, and $A \subseteq B \subseteq \bar{A}$, then $B$ is also connected.*

*Proof.* Let $X$ be a space, let $A$ be a connected subspace of $X$, and let $B$ be a set contained in the closure of $A$ that also contains $A$ itself. For the sake of contradiction, we will assume that $B$ is not connected. If that is the case, then we can write $B$ as the disjoint union of two nonempty sets. Denote the separation of $B$ as the disjoint union of $C$ and $D$, where $C$ and $D$ are disjoint, nonempty, open subsets. Because $A$ is connected, there are two possibilities. The first possibility is that $A$ is a subset of $C$. It is also possible that $A$ is a subset of $D$. Without loss of generality, suppose that $A$ is a subset of $C$. If $A$ is a subset of $C$, and $D$ is nonempty, we can exhibit an element $b$ in $B$ that is also in $D$. Theorem 17.5 of "Topology" by Munkres states the following:

**Theorem 2** (17.5 of Munkres)**.** *Let $A$ be a subset of the topological space $X$.*

1. *Then $x \in \bar{A}$ if and only if every open set $U$ containing $x$ intersects $A$.*

2. *Supposing the topology of $X$ is given by a basis, then $x \in \bar{A}$ if and only if every basis element $B$ containing $x$ intersects $A$.*

Since the element $b$ in $B$ is also in the closure of $A$, we know that by the abovementioned theorem, every open set containing $b$ (of which $D$ is an example) intersects $A$. This contradicts the fact that the sets $C$ and $D$ are disjoint. Because of this, no such separation of $B$ as disjoint, nonempty, open subsets $C$ and $D$ could exist. $\square$

Please keep this focus in mind as you work on this week's problems:

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1. Let $X = \mathbb{R}^2$ with the dictionary order topology. Is $X$ connected? Why, or why not?

2. Let $X$ be an ordered set, and give it the order topology. Assume that $X$ is connected, and show that $X$ is a linear continuum.

3. Let $p : X \to Y$ be a quotient map, and assume $Y$ to be connected. Assume further that for each point $y \in Y$, the subspace $p^{-1}(\{y\}) \subseteq X$ (called the *fibre* over $y$) is connected (when given the subspace topology). Show that $X$ is connected. Is the converse true? That is, if $X$ is connected, must its fibres be, too?

4. Define $C$ to be the *Cantor set*: this is obtained by iteratively removing the middle third of every subinterval of $[0, 1]$. More carefully, let $A_0 := [0, 1]$, and inductively define

$$A_n := A_{n-1} \setminus \left( \bigcup_{k=0}^{3^{n-1}-1} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \right)$$

(this is not a maximally efficient presentation of $A_n$; there are some intervals being taken out of $A_{n-1}$ in this definition that are already missing from $A_{n-1}$, e.g., any inside of $(\frac{1}{3}, \frac{2}{3})$ if $n - 1 > 0$.). Now set

$$C = \bigcap_{n=0}^{\infty} A_n.$$

(a) Show that $C$ is *totally disconnected* – its only connected subsets are singletons.

(b) Show, however, that $C$ has no *isolated points*: there are no points $c \in C$ with $\{c\}$ being open in $C$.

**Hint:** It may be helpful to first show that $A_n$ is a disjoint union of closed intervals of length $\frac{1}{3^n}$, and the endpoints of these intervals lie in $C$.