Homework #5 for MATH 5345H: Introduction to Topology

October 27, 2018

Due Date: Friday 2 November in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills. This week, I would like you to focus on the structure of sentences within your proof. Strong proofs are written in paragraphs that comprise complete sentences. Here are some qualities that sentences in mathematical writing should have. Sentences should:

1. start with a capital letter (and in particular, a word and not a variable),
2. end with with a period,
3. contain a subject and a verb, and
4. comprise words instead of symbols.

On the last point: certainly symbols do belong in mathematical writing:

> "Let $f : X \to Y$ be a continuous map."

is a perfectly valid mathematical sentence. In contrast,

> "$\forall x, \exists y$ s.t. $y > x$"

could certainly be improved to "For all x, there exists a y such that $y > x$" if it occurs in the midst of an argument. On the other hand, you're perfectly welcome to use such shorthand in defining sets, for instance. For example,

$$S := \{f : \mathbb{R} \to \mathbb{R} \mid \exists N \, s.t. \, \forall x, f(x) < N\}$$

describes the set of bounded-above functions on $\mathbb{R}$. Getting this right can be difficult, but the theme is certainly: if you're writing an explanation, please make it a sentence.

Here are some examples of statements from proofs written in 5XXX-level math courses that **do not** constitute proper sentences. Can you identify why each of these statements do not belong in a mathematical proof?

- 109 is a factor! $109 - 1 = 108 = 2^2 3^2$ is 3-smooth.
- $\implies y$ is a nonsquare
- group of T persons $a_1, \dots, a_T$ = heights of T persons
- $Y = \#$ nondefective “”
- goal: find $P(X_2 = 1 | X_1 = 1)$
- $\therefore$ independent

Please keep this in mind as you approach this week's homework:

1. Let $X = \mathbb{R}^n$ with its standard topology, and let $\sim$ be the equivalence relation on X where $x \sim y$ if and only if there is an orthogonal[1] $n \times n$ matrix $A = (a_{ij})$ with $Ax = y$.

   (a) Show that $\sim$ is an equivalence relation.

   (b) Let $x = (1, 0, \dots, 0)$. Describe the equivalence class $[x] \subseteq X$ geometrically.

   (c) Let $f : X \to \mathbb{R}_{\geq 0}$ be the function $f(x) = |x|$ which takes the norm of a vector. Show that f is continuous (when $\mathbb{R}_{\geq 0}$ is given the subspace topology of $\mathbb{R}$) and surjective.

   (d) Using f, construct a continuous bijection $(X/\sim) \to \mathbb{R}_{\geq 0}$.

   (e) Let $g : \mathbb{R}_{\geq 0} \to X$ be defined by $g(t) = (t, 0, \dots, 0)$. Show that g and $q \circ g$ are continuous, where $q : X \to (X/\sim)$ is the quotient map.

   (f) Using parts (d) and (e), show that the quotient space $(X/\sim)$ is homeomorphic to $\mathbb{R}_{\geq 0}$.

2. Show that $\mathbb{R} \times \mathbb{R}$ with the dictionary order topology is metrizable.

3. Let $X \subseteq \mathbb{R}^\omega$ be the set

$$X := \{(x_1, x_2, \dots) \mid \exists N > 0 \text{ such that } x_n = 0 \text{ if } n > N\}.$$

   What is $\overline{X}$ when $\mathbb{R}^\omega$ is given the product topology? What about the box topology?

[1] That is, the rows and columns of A form an orthonormal basis of $\mathbb{R}^n$; equivalently, $A^t A = I = AA^t$.