Midterm for MATH 5345H: Introduction to Topology

October 16, 2017

Due Date: Monday 23 October **in class**.

You may use your book, notes, and old homeworks for this exam. When using results from any of these sources, please cite the result being used. Please explain all of your arguments carefully.

Please do not communicate with other students about the exam. You are free to contact me with questions about the exam at any time.

1. A subset Z of a topological space X is called *dense* in X if $\overline{Z} = X$. If $f : X \to Y$ is continuous and surjective, and $Z \subseteq X$ is dense in X, show that $f(Z)$ is dense in Y.

2. Let X be a set and let $d : X \times X \to \mathbb{R}$ be a function which satisfies conditions (2) and (3) in the definition of a metric (Munkres, p. 119), but which satisfies only the following relaxed version of condition (1):

   $d(x, y) \geq 0$ for all $x, y \in X$, but we *can* have $d(x, y) = 0$ even when $x \neq y$.

   (a) We say that $x \sim y$ if $d(x, y) = 0$. Show that $\sim$ is an equivalence relation.

   (b) If $x \sim y$ and $z \in X$, show that $d(x, z) = d(y, z)$.

   (c) Let $X/\sim$ be the set of equivalence classes for $\sim$, and define a function

   $$d' : (X/\sim) \times (X/\sim) \to \mathbb{R}$$

   as follows: if $E_0, E_1 \in X/\sim$ are equivalence classes, choose $x \in E_0$ and $y \in E_1$, and set

   $$d'(E_0, E_1) = d(x, y).$$

   Deduce from part (b) that d' is well-defined. Show that d' is really a metric: it satisfies conditions (1), (2) and (3) of the definition on p. 119.

3. Let $X = [0, 2\pi]$, and define $f : X \to S^1$ by the formula

   $$f(t) = e^{it}.$$

   (a) Let $\sim$ be the equivalence relation on X given by

   $$\forall x \in X,\ x \sim x. \text{ Also } 0 \sim 2\pi, \text{and } 2\pi \sim 0.$$

   Let $q : X \to X/\sim$ be the quotient map. Show that there is a homeomorphism $\overline{f} : X/\sim \to S^1$ which satisfies $\overline{f} \circ q = f$.

(b) Let g be the restriction of f to $[0, 2\pi)$. Show that g is continuous and bijective. Is g a homeomorphism? Prove your answer.

4. Let X be a totally ordered set.

   (a) Give X the order topology T_{order}; show that (X, T_{order}) is Hausdorff.

   (b) For an element $a \in X$, define the *left ray* to be:

   $$(-\infty, a) := \{x \in X \mid x < a\}$$

   Define B_{left} to consist of left rays, along with the empty set and X.

   $$B_{left} = \{(-\infty, a) \mid a \in X\} \cup \{\emptyset, X\}$$

   Show that B_{left} is a basis for a topology on X, the *left order topology*, T_{left}.

   (c) If X has the least upper bound property (Munkres, pg. 27), show that in fact B_{left} is not just a basis, but a topology. Equivalently, $B_{left} = T_{left}$.

   (d) Show that if X has at least two elements and the least upper bound property, then T_{order} is strictly finer than T_{left}.

   (e) Assume that X has at least two elements, and give it the left order topology. Show that (X, T_{left}) is not Hausdorff.