Homework #7 for MATH 5345H: Introduction to Topology

November 1, 2017

Due Date: Monday 6 November in class.

1. Define C to be the Cantor set: this is obtained by iteratively removing the middle third of every subinterval of $[0,1]$. More carefully, let $A_0 := [0,1]$, and inductively define
$$A_n := A_{n-1} \setminus \left(\bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \right)$$
(this is not a maximally efficient presentation of A_n; there are some intervals being taken out of A_{n-1} in this definition that are already missing from A_{n-1}, e.g., any inside of $(\frac{1}{3}, \frac{2}{3})$ if $n-1>0$.). Now set
$$C = \bigcap_{n=0}^{\infty} A_n.$$
   (a) Show that C is compact.
   (b) Show that C is *totally disconnected* – its only connected subsets are singletons.
   (c) Show, however, that C has no *isolated points*: there are no points $c \in C$ with $\{c\}$ being open in C.
   Hint: It may be helpful to first show that A_n is a disjoint union of closed intervals of length $\frac{1}{3^n}$, and the endpoints of these intervals lie in C.

2. Let X be a metric space. Recall that X is *complete* if every Cauchy sequence in X converges (recall further that $\{x_n\}$ is *Cauchy* if, for every $\epsilon > 0$, there exists an N such that if $m, n > N$, $d(x_m, x_n) < \epsilon$).

   Let X be a complete metric space and let $f : X \to X$ be a function satisfying the following condition: there is some $r \in [0,1)$ such that for any points $x, y \in X$,
$$d(f(x), f(y)) \leq r d(x, y).$$
   Such a map f is called a *contraction mapping*.

(a) Show that, for any $x \in X$, the sequence

$$x, f(x), f(f(x)), \cdots$$

is Cauchy.

(b) (The contraction mapping theorem) Using the previous problem, show that f has a unique fixed point (that is, there is a unique $x_0 \in X$ such that $f(x_0) = x_0$).

(c) Give an example to show that the conclusion of the contraction mapping theorem doesn't necessarily hold when

$$d(f(x), f(y)) = d(x, y),$$

(such a map f is called an *isometry*), even if X is compact.

(d) Suppose that X is a compact metric space and $f : X \to X$ is an isometry. Show that f is surjective.

Hint: consider once again the sequence

$$x, f(x), f(f(x)), \cdots$$

this time using sequential compactness to find a sequence of points of $\mathrm{Im}(f)$ converging to x.

(e) Using the previous problem, show that an isometry is a homeomorphism.