Homework #5 for MATH 5345H: Introduction to Topology

October 2, 2017

Due Date: Monday 9 October in class.

1. Endow $\mathbb{R}$ with its usual topology.

   (a) Show that the sum and product functions

   $$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ f(x, y) = x + y$$

   and

   $$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ g(x, y) = xy$$

   are continuous.

   (b) Let $\overline{\mathbb{R}}$ be the topological space whose underlying set is $\mathbb{R}$ and whose open sets are $\emptyset$, $\mathbb{R}$ and the rays

   $$(a, \infty) = \{r \in \mathbb{R} \mid r > a\}$$

   for each $a \in \mathbb{R}$. Are the sum and product functions

   $$\overline{f} : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}},\ \overline{f}(x, y) = x + y$$

   and

   $$g : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}},\ \overline{g}(x, y) = xy$$

   continuous? Prove or disprove.

   In this question it's OK to be informal about whether a subset of $\mathbb{R}^2$ is open.

2. A continuous map $f : X \to Y$ is called an *open map* if for every open set $U \subseteq X$, its image $f(U) \subseteq Y$ is open. Similarly, f is called a *closed map* if for every closed set $Z \subseteq X$, its image $f(Z) \subseteq Y$ is closed.

   Suppose K is a subset of X, equipped with its subspace topology, and let $i : K \to X$ be the inclusion of K into X. Show that i is an open map if and only if K is an open subset of X, and that i is a closed map if and only if K is a closed subset of X.

3. Let X be a topological space, and let $\Delta \subseteq X \times X$ be the *diagonal*:

$$\Delta := \{(x, x) \mid x \in X\} \subseteq X \times X$$

Show that X is Hausdorff if and only if Δ is a closed subset of $X \times X$.