# Homework #4 for MATH 5345H: Introduction to Topology

September 26, 2017

**Due Date:** Monday 2 October in class.

1. Let $X = \mathbb{Z}_{\geq 0} \times [0,1)$. Both $\mathbb{Z}_{\geq 0}$ and $[0,1)$ are ordered sets, coming from the order on $\mathbb{R}$, by restriction to these subsets. Equip $X$ with the dictionary order, and the order topology coming from this order. Let $Y = [0,\infty) \subseteq \mathbb{R}$ be the closed ray in $\mathbb{R}$, equipped with the subspace topology. This question aims to show that $X$ and $Y$ are homeomorphic.

   (a) Define $f : X \to Y$ by the formula $f(n,t) = n + t$. Define $g : Y \to X$ by $g(x) = (\lfloor x \rfloor, x - \lfloor x \rfloor)$, where $\lfloor x \rfloor$ is the greatest integer less than $x$. Verify that these are mutually inverse bijections.

   (b) Show that $f$ and $g$ are continuous, and conclude that $X$ and $Y$ are homeomorphic.

2. Show that the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ with the discrete topology.

3. Show that the collection

$$\{(a,b) \times (c,d) \mid a,b,c,d, \in \mathbb{Q},\ a < b,\ c < d\}$$

   is a countable basis for the product topology on $\mathbb{R}^2$.

4. Let $X$ be an ordered set, and give it the order topology. Show that the closure $\overline{(a,b)}$ of the open interval $(a,b)$ is contained in the closed interval $[a,b]$; that is, $\overline{(a,b)} \subseteq [a,b]$. Under what conditions (i.e., for what properties of the order) are they equal?

5. Show that every order topology is Hausdorff.