1^{st} midterm for MATH 1272: Calculus II, section 030

Name:

ID #:

Section Number:

Teaching Assistant:

Instructions:

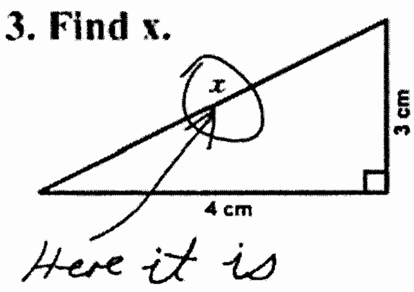
- *Please don't turn over this page until you are directed to begin.*
- There are 6 problems (and problems 1, 2, 4, and 5 have 2 parts) on this exam.
- There are 7 pages to the exam, including this page. All of them are one-sided. If you run out of room on the page that you're working on, use the back of the page.
- Please show all your work. Answers unsupported by an argument will get little credit.
- Scientific calculators are allowed. No books or notes are allowed. Please turn off your cell phones.

Grading summary

Problem:	1	2	3	4	5	6	total
Possible:	10 points	10 points	10 points	15 points	15 points	10 points	70 points
Grade:							

Some helpful formulas

$\sin^2(x)+\cos^2(x)=1$	$\tan^2(x)+1=\sec^2(x)$	$1+\cot^2(x)=\csc^2(x)$
$2\sin^2(x)=1-\cos(2x)$	$2\cos^2(x)=1+\cos(2x)$	$2\sin(x)\cos(x)=\sin(2x)$
$\int\tan(x)\,dx=\ln\lvert\sec(x)\rvert+C$	$\int\sec(x)\,dx=\ln\lvert\sec(x)+\tan(x)\rvert+C$	$\int\csc(x)\,dx=\ln\lvert\csc(x)-\cot(x)\rvert+C$



1. **(10 points total, 5 points each)**

(a) Evaluate the integral

$$\int x\sec^2(x)\,dx.$$

Integrate by parts, with $u = x$, $du = dx$, $dv = \sec^2 x\,dx$, $v = \tan x$

$$= uv - \int v\,du = x\tan x - \int \tan x\,dx$$
$$= x\tan x - \ln|\sec(x)| + C$$

(b) Evaluate the integral

$$\int x\tan^2(x)\,dx.$$

Hint: The previous part may be helpful.

Since $\tan^2(x)+1 = \sec^2 x$, we have

$$\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx$$
$$= \int x\sec^2 x\,dx - \int x\,dx$$
$$= x\tan x - \ln|\sec(x)| - \frac{x^2}{2} + C$$

2. **(10 points total, 5 points each)** Determine if the following integrals are convergent or divergent. If convergent, evaluate the integral:

(a)

$$\int_5^\infty \frac{e^x+7}{e^x+2}\,dx.$$

Since $\dfrac{e^x+7}{e^x+2} > 1$, $\displaystyle\int_5^\infty \frac{e^x+7}{e^x+2}\,dx > \int_5^\infty 1\,dx$.

Since the latter diverges, so too must the former.

(b)

$$\int_0^\infty \frac{1}{x(\ln x)^3}\,dx.$$

A better question would have been $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\,dx$, which is computed by substitution:

$u = \ln x$, $du = \frac{1}{x}\,dx$, so

$$= \int \frac{du}{u^3} = \frac{-1}{2u^2}$$

$$= \left.\frac{-1}{2(\ln x)^2}\right]_2^\infty$$

$$= \lim_{n\to\infty}\left(\frac{-1}{2(\ln n)^2} + \frac{1}{2(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}$$

However: the original integral diverges, because of the pole at $x=1$ for $\frac{1}{\ln x}$:

$$\int_1^2 \frac{1}{x(\ln x)^3}\,dx = \lim_{n\to 1}\left(\frac{-1}{2(\ln n)^2} + \frac{1}{2(\ln 2)^2}\right) = -\infty$$

Since the integral on this subinterval diverges, it diverges on $\int_0^\infty$, too.

3. **(10 points)** Evaluate the integral

$$\int \frac{dx}{x\sqrt{5-x^2}}.$$

Use the trig. substitution $x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\, d\theta$:

$$\int \frac{dx}{x\sqrt{5-x^2}} = \int \frac{\sqrt{5}\cos\theta\, d\theta}{\sqrt{5}\sin\theta\sqrt{5-5\sin^2\theta}}$$

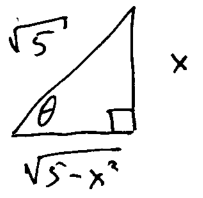
$$= \int \frac{\cos\theta\, d\theta}{\sqrt{5}\sin\theta\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta\, d\theta}{\sqrt{5}\sin\theta\cos\theta}$$

$$= \frac{1}{\sqrt{5}}\int \frac{d\theta}{\sin\theta} = \frac{1}{\sqrt{5}}\int \csc\theta\, d\theta$$

$$= \frac{1}{\sqrt{5}} \ln|\csc\theta - \cot\theta| + C$$

To rewrite in terms of x:



$$\frac{x}{\sqrt{5}} = \sin\theta$$

$$\Rightarrow \quad \csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{5}}{x}$$

and $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{x}{\sqrt{5-x^2}}$

so:

$$\int \frac{dx}{x\sqrt{5-x^2}} = \frac{1}{\sqrt{5}} \ln\left| \frac{\sqrt{5}}{x} - \frac{x}{\sqrt{5-x^2}} \right| + C$$

4. **(15 points total)**

(a) **(10 points)** Evaluate the integral

$$\int \frac{2u\,du}{u^2+2u-3}.$$

(u^2+2u-3)
$= (u+3)(u-1)$

We use partial fractions:

$$\frac{2u}{u^2+2u-3} = \frac{A}{u+3} + \frac{B}{u-1}$$

$$2u = A(u-1) + B(u+3) = Au - A + Bu + 3B$$
$$= (A+B)u + (3B-A)$$

So $A+B=2$, $3B-A=0 \Rightarrow A=3B \Rightarrow 4B=2 \Rightarrow B=\frac{1}{2} \Rightarrow A=\frac{3}{2}$

$$\int \frac{2u\,du}{u^2+2u-3} = \frac{3}{2}\int \frac{du}{u+3} + \frac{1}{2}\int \frac{du}{u-1} = \boxed{\frac{3}{2}\ln|u+3| + \frac{1}{2}\ln|u-1| + C}$$

(b) **(5 points)** Evaluate the integral

$$\int \frac{dx}{2\sqrt{x+3}+x}.$$

Hint: The previous part may be helpful.

Set $u = \sqrt{x+3}$. Then $u^2 = x+3$, so:

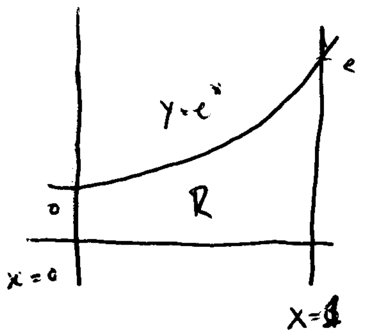
$x = u^2 - 3$ and $dx = 2u\,du$.

Then
$$\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2u\,du}{2u+u^2-3}$$
$$= \frac{3}{2}\ln|u+3| + \frac{1}{2}\ln|u-1| + C$$
$$= \frac{3}{2}\ln\left|\sqrt{x+3}+3\right| + \frac{1}{2}\ln\left|\sqrt{x+3}-1\right| + C$$

5. **(15 points total)**

(a) **(5 points)** Find the area of the region $\mathcal{R}$ in the plane bounded by the curves

$$y = e^{x}; \quad y = 0; \quad x = 0; \quad x = 1.$$



$$A = \int_0^1 e^x\,dx = e^x\Big]_0^1 = e^1 - e^0$$

$$= e - 1$$

(b) **(10 points)** Find the centroid (or center of mass) of the region $\mathcal{R}$.

The centroid is the point $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{1}{A}\int_0^b x f(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx$$

In our case,

$$\bar{x} = \frac{1}{e-1}\int_0^1 x e^x\,dx \qquad \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix}$$

$$= \frac{1}{e-1}\left(xe^x - \int_0^1 e^x\,dx\right)$$

$$= \frac{1}{e-1}\left[xe^x - e^x\right]_0^1$$

$$= \frac{1}{e-1}\left[(e-e) - (0-1)\right] = \frac{1}{e-1}$$

and

$$\bar{y} = \frac{1}{e-1}\int_0^1 \frac{1}{2}(e^x)^2\,dx = \frac{1}{2(e-1)}\int_0^1 e^{2x}\,dx = \frac{1}{2(e-1)}\left[\frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{1}{2(e-1)}\left(\frac{e^2}{2} - \frac{1}{2}\right) = \frac{1}{4(e-1)}(e^2-1) = \frac{e+1}{4}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right)$$

6. **(10 points)** Find the length of the curve $y = \ln(\cos(x))$ on the interval $0 \le x \le \pi/4$.

Recall that arclength is computed as:

$$L = \int_0^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For $y = \ln(\cos(x))$, we have

$$\frac{dy}{dx} = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan x, \text{ so:}$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} |\sec x|\, dx = \ln|\sec x + \tan x| \Big]_0^{\pi/4}$$

$$= \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

$$= \ln|\sqrt{2} + 1|$$

$$= \ln(\sqrt{2} + 1)$$

2^{nd} midterm for MATH 1272: Calculus II, section 030

Name: **Section Number:**
ID #: **Teaching Assistant:**

Instructions:

- *Please don't turn over this page until you are directed to begin.*
- There are 6 problems (and problems 1, 3, 5, and 6 have multiple parts) on this exam.
- There are 7 pages to the exam, including this page. All of them are one-sided. If you run out of room on the page that you're working on, use the back of the page.
- Please show all your work. Answers unsupported by an argument will get little credit.
- Scientific calculators are allowed. No books or notes are allowed. Please turn off your cell phones.

Grading summary

Problem:	1	2	3	4	5	6	total
Possible:	10 points	10 points	15 points	10 points	15 points	20 points	80 points
Grade:							

Some helpful formulas

$\sin^2(x)+\cos^2(x)=1$	$\tan^2(x)+1=\sec^2(x)$	$1+\cot^2(x)=\csc^2(x)$
$2\sin^2(x)=1-\cos(2x)$	$2\cos^2(x)=1+\cos(2x)$	$2\sin(x)\cos(x)=\sin(2x)$
$\int\tan(x)\,dx=\ln\lvert\sec(x)\rvert+C$	$\int\sec(x)\,dx=\ln\lvert\sec(x)+\tan(x)\rvert+C$	$\int\csc(x)\,dx=\ln\lvert\csc(x)-\cot(x)\rvert+C$

1. **(10 points total, 5 points each)** Consider the differential equation

$$y' = x + y^2.$$

(a) Sketch a direction field for this differential equation in the region $-2 \le x \le 2$ and $-2 \le y \le 2$.

Notice that

$0 = y' = x + y^2$

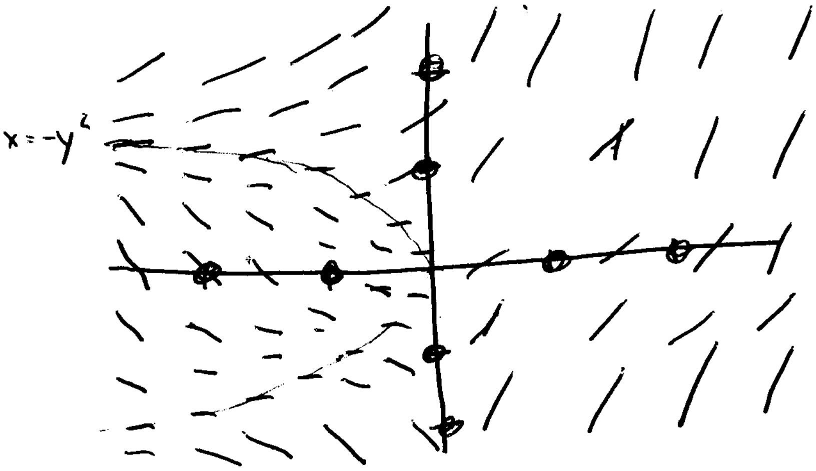
if $x = -y^2$

Further:

$y' > 0$ if $x > -y^2$

and

$y' < 0$ if $x < -y^2$



(b) Use Euler's method with step size 0.2 to estimate $y(0.4)$ where $y(x)$ is the solution of the differential equation with initial value $y(0) = 0$.

Euler's method w/ initial values x_0, y_0 for $y' = f(x,y)$ and step size h is:

$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$ and $x_n = x_{n-1} + h$

So: $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$ and

$y_0 = y(0) = 0$

$y_1 = y_0 + h f(x_0, y_0) = 0 + 0.2(x_0 + y_0^2)$

$= 0 + 0.2(0 + 0^2) = 0$

$y_2 = y_1 + h f(x_1, y_1) = 0 + 0.2(0.2 + 0^2) = 0.04$

so $y(0.4) \approx 0.04$

2. **(10 points)** Solve the differential equation

$$e^{-y}y' + \cos x = 0$$

with the initial condition $y(0) = 1$.

$$e^{-y} y' + \cos x = 0$$

$$e^{-y} \frac{dy}{dx} = -\cos x$$

$$\int e^{-y}\,dy = \int -\cos x\,dx$$

$$-e^{-y} = -\sin x + C$$

$$e^{-y} = +\sin x + D$$

$$-y = \ln(+\sin x + D)$$

$$y = -\ln(+\sin x + D)$$

$$1 = y(0) = -\ln(+\sin 0 + D)$$

$$= -\ln(D)$$

$$-1 = \ln(D)$$

$$D = 1/e$$

So:

$$\cancel{y = (-\ln(\pm\sin x \pm 1/e))}$$

$$\boxed{y = -\ln(\sin x + 1/e)}$$

3. **(15 points total, 5 points each)** Consider a population of rabbits whose initial population is $P(0) = 50$, and whose population $P(t)$ at time t (in years) satisfies the differential equation

$$\frac{dP}{dt} = 0.4P - 0.001P^2$$

(a) What is the carrying capacity of the population?

In the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ $= kP - \frac{kP^2}{M}$

M represents the carrying capacity. So

$k = 0.4$ and $\frac{k}{M} = +0.001$

So $M = \frac{0.4}{0.001} = 400$

(b) What is $P'(0)$?

$$P'(0) = \frac{dP}{dt}(0) = 0.4 \cdot P(0) - 0.001(P(0))^2$$
$$= (0.4)(50) - 0.001(50^2)$$
$$= 20 - \frac{2500}{1000} = 20 - 2.5 = \boxed{17.5}$$

(c) When will the population reach half the carrying capacity?

The solution to the logistic eq'n is

$$P(t) = \frac{M}{1 + Ae^{-kt}} \quad \text{where } A = \frac{M}{P_0} - 1 = \frac{400}{50} - 1 = 7$$

so: we need to solve for t in: $P(t) = \frac{1}{2}M$:

$$\frac{1}{2}M = \frac{M}{1 + Ae^{-kt}} \iff 1 + Ae^{-kt} = 2$$
$$Ae^{-kt} = 1$$
$$e^{-kt} = 1/7$$
$$-kt = \ln(1/7) = -\ln 7$$
$$t = \frac{\ln 7}{k} = \boxed{\frac{\ln 7}{0.4}}$$

4. **(10 points)** Solve the differential equation

$$(1+s)\frac{dv}{ds} + v = 1 + s$$

where $s > 0$, and $v(0) = 5$.

Notice that the left side is already in form for integration. That is:

$\frac{d}{ds}\left((1+s)v\right) = (1+s)\frac{dv}{ds} + v$, solve DE is:

$$\frac{d}{ds}\left((1+s)v\right) = 1+s$$

$$\Downarrow$$

$$\int \frac{d}{ds}\left((1+s)v\right) ds = \int 1+s \, ds$$

$$(1+s)v = s + \frac{s^2}{2} + C$$

$$v = \frac{s + s^2/2 + C}{1+s}$$

$$5 = v(0) = \frac{0 + 0^2/2 + C}{1+0} = C$$

So:

$$\boxed{v = \frac{s + s^2/2 + 5}{1+s}}$$

5. **(15 points total, 5 points each)** Populations of birds and insects are modelled by the differential equations

$$\frac{dx}{dt} = 4x - 0.2xy \quad \text{and} \quad \frac{dy}{dt} = -2y + 0.08xy$$

(a) Which of the variables, x or y, represents the bird population and which represents the insect population? Explain why.

x represents the insects and y represents the birds: The interaction terms (involving xy) are negative in dx/dt, indicating that the x population decreases with interactions. Similarly, they are positive in dy/dt, so y increases with interactions. Thus y must be the birds ~~eat~~ which increase whenever they meet ~~eat~~ insect (whose population decreases).

(b) Find the equilibrium solutions, and explain their meaning.

The equilibrium solutions occur when

$0 = \frac{dx}{dt} = 4x - 0.2xy$ and $0 = \frac{dy}{dt} = -2y + 0.08xy$

$4x = 0.2xy$ $\qquad$ $2y = 0.08xy$

So, $x = 0$ or $y = 20$ $\qquad$ $y = 0$ or $x = 2/0.08 = 25$

Two equilibria: $(0,0)$: everything has died.

$(25, 20)$: exactly 25 insects and 20 birds balances the populations perfectly

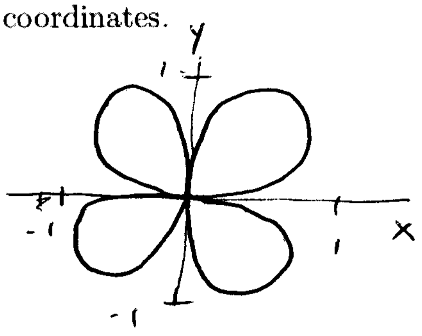
(c) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2y + 0.08xy}{4x - 0.2xy}$$

6. **(20 points total, 5 points each)** Consider the curve in polar coordinates (r, θ) defined by the equation $r = \sin(2\theta)$.

(a) Sketch a graph of the curve in Cartesian (i.e., (x, y)) coordinates. **Note:** you do *not* need to find a *formula* for the curve in Cartesian coordinates.

Notice that for $\theta \in [0, 2\pi]$, $r = 0$ whenever $\theta = 0, \pi/2, \pi, 3\pi/2$, and $r \geq 0$ on $[0, \pi/2]$, $[\pi, 3\pi/2]$
$r \leq 0$ on $[\pi/2, \pi]$, $[3\pi/2, 2\pi]$
Finally $|r| \leq 1$



(b) At what values of θ with $0 \leq \theta \leq 2\pi$ does the curve intersect itself?

$\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$. (just when $r = 0$)

(c) Find the area of one loop of the curve (that is, between two points of intersection).

The first intersection is at $\theta = \pi/2$. So the area is

$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \tfrac{1}{2} \sin^2(2\theta) \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} 1 - \cos(4\theta) \, d\theta = \frac{1}{4} \left[\theta - \frac{\sin(4\theta)}{4} \right]_0^{\pi/2}$$

$$= \frac{1}{4} \left[(\pi/2 - 0) - (0 - 0) \right] = \pi/8$$

(d) Find the slope of the curve corresponding to the point $\theta = \pi/4$.

We want $\frac{dy}{dx}$. But $x = r\cos\theta$ $y = r\sin\theta$

$x = \sin(2\theta)\cos\theta$ $y = \sin(2\theta)\sin\theta$

at $\theta = \pi/4$

so $\frac{dx}{d\theta} = 2\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta = 0 - 1 \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$

$\frac{dy}{d\theta} = 2\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta = 0 + 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

So $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = \boxed{-1}$

3^{rd} midterm for MATH 1272: Calculus II, section 030

Name: **Section Number:**
ID #: **Teaching Assistant:**
Instructions:

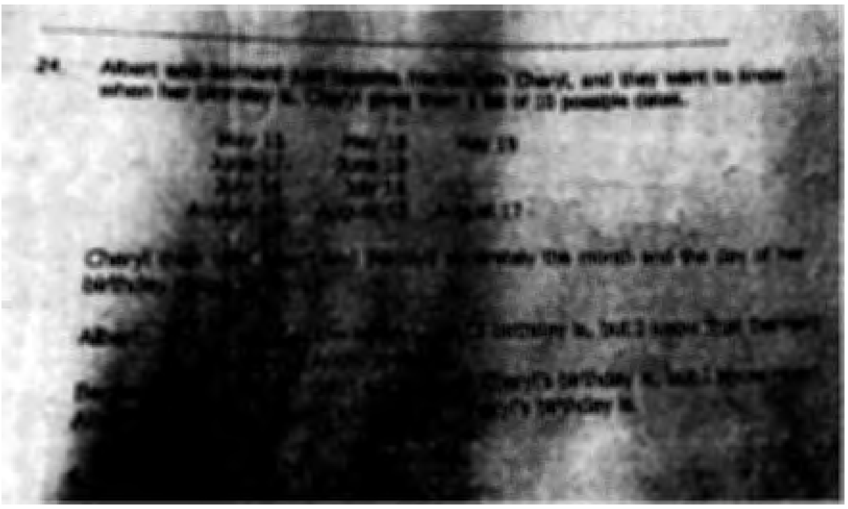
- *Please don't turn over this page until you are directed to begin.*
- There are 5 problems on this exam, and all except problem 2 have multiple parts.
- There are 7 pages to the exam, including this page. All of them are one-sided. If you run out of room on the page that you're working on, use the back of the page.
- Please show all your work. Answers unsupported by an argument will get little credit.
- Scientific calculators are allowed. No books or notes are allowed. Please turn off your cell phones.

Grading summary

Problem:	1	2	3	4	5	total
Possible:	10 points	5 points	30 points	15 points	20 points	80 points
Grade:						

Some helpful formulas

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$
$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$



1. **(10 points total, 5 points each)** Determine whether the following sequences $\{a_n\}$ converge or diverge. If they converge, compute the limit, $\lim_{n\to\infty} a_n$.

(a) $a_n = \dfrac{2+7n^2}{n+n^2}$.

$$\lim_{n\to\infty} \frac{2+7n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2(7+2/n^2)}{n^2(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{(7+2/n^2)}{1+1/n} = 7 \quad \text{(converges)}$$

(b) $a_n = \dfrac{n^2}{\sqrt{n^2-n}}$.

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{1-1/n}}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{1-1/n}}$$

This diverges; the numerator diverges, but the denom. converges to 1.

2. **(5 points)** Consider the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n \cdot 5^n}$. How many terms of the series do we need to sum in order to be within an error of at most 10^{-4} of the actual infinite sum?

The alternating series remainder estimate is:

$|R_n| < \dfrac{1}{(n+1)5^{n+1}}$. So, for $|R_n| < 10^{-4}$, we need

$$\frac{1}{(n+1)5^{n+1}} < \frac{1}{10^4} \Leftrightarrow (n+1)5^{n+1} > 10^4 = 5^4 2^4$$

$$(n+1)5^{(n-3)} > 16$$

$\underline{n=4}$ works; $(4+1)5^{(4-3)} = 25 > 16$.

3. **(30 points total, 5 points each)** Are the following series absolutely convergent, conditionally convergent, or divergent? Justify for your answer. If they converge, you do *not* need to compute their sum.

(a) $\sum_{n=2}^{\infty}(-1)^n\left(\frac{n^3}{n^4-1}\right)$.

Converges by the alternating series test, since $\frac{n^3}{n^4-1}$ is eventually decreasing, with limit $=0$. However, it is not absolutely convergent, since

$$\left|(-1)^n\left(\frac{n^3}{n^4-1}\right)\right| = \frac{n^3}{n^4-1} > \frac{n^3}{n^4} = 1/n,$$ so

$$\sum_{n=2}^{\infty}\left|(-1)^n\left(\frac{n^3}{n^4-1}\right)\right| > \sum_{n=2}^{\infty}\frac{1}{n}$$ which diverges (harmonic)

conditional convergence

(b) $\sum_{n=1}^{\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right)$.

This diverges, since

$$\lim_{n\to\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right) = \lim_{n\to\infty}\ln\left(\frac{n^2(1+1/n^2)}{n^2(2+1/n^2)}\right)$$

$$= \lim_{n\to\infty}\ln\left(\frac{1+1/n^2}{2+1/n^2}\right) = \ln(1/2) \neq 0$$

Since the <u>sequence</u> does not converge to 0, then <u>series</u> cannot converge

(c) $\sum_{n=2}^{\infty}\left(\frac{-n}{2n+1}\right)^{5n}$.

Use the root test:

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-n}{2n+1}\right)^{5n}\right|} = \lim_{n\to\infty}\left(\frac{n}{2n+1}\right)^5$$

$$= \lim_{n\to\infty}\left(\frac{1}{2+1/n}\right)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32} < 1$$

So the series is absolutely convergent

(Continued from previous page)

(d) $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)$. $\geq \sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty}\frac{1}{n}$] harmonic series

so ↑ is larger than a divergent series, and must therefore diverge (comparison test)

(e) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$. Use the integral test:

$$\int_2^{\infty}\frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln x|\Big]_2^{\infty}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \underbrace{\lim_{n\to\infty} \ln|\ln(n)|}_{\text{diverges, so the sum also diverges.}} - \ln|\ln(2)|$$

diverges, so the sum also diverges.

(f) $\sum_{n=1}^{\infty}\frac{3^n\cdot n^2}{n!}$. Use the Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2}$$

$$= \lim_{n\to\infty}\frac{3}{n+1}\left(\frac{n+1}{n}\right)^2 = 0$$, since converges to 1

and converges to 0. So by the Ratio Test, this converges (absolutely)

4. **(15 points total, 5 points each)** Find the radius of convergence of the following power series:

(a) $\sum_{n=1}^{\infty}(-4)^n(x-5)^n$. The Root test says this converges

if $\lim_{n\to\infty} \sqrt[n]{|(-4)^n(x-5)^n|} < 1$

$= \lim_{n\to\infty} 4|x-5| < 1$

$|x-5| < 1/4$ so radius $= 1/4$

(b) $\sum_{n=1}^{\infty} n^n x^n$. Again, by the root test:

$1 > \lim_{n\to\infty} \sqrt[n]{|n^n x^n|} = \lim_{n\to\infty} n|x|$

This limit is 0 if $x = 0$, and ~~either~~ $+\infty$ if $x \neq 0$, Therefore the series converges only at $x = 0$, so $r = 0$.

(c) $\sum_{n=1}^{\infty} \frac{(x-7)^n}{n!}$. Ratio test:

$1 > \lim_{n\to\infty} \frac{|x-7|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x-7|^n} = \lim_{n\to\infty} \frac{|x-7|}{n+1} = 0$

The limit is 0 (<1) for every x, so the ~~seat~~ series converges for every x. thus Radius $= \infty$

5. **(20 points total, 5 points each)** Find a power series representation for the following functions and determine the radius of convergence:

Radius = ∞

(a) $f(x) = x\cos x$, centered at $a = 0$.

since $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, we have ✓

$$x\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

This has the same radius of convergence as cos x: $R = \infty$

(b) $f(x) = \cos x$, centered at $a = \pi$.

$f^{(1)}x = -\sin x$

$f^{(2)}x = -\cos x$

$f^{(3)}x = \sin x$

$f^{(4)}x = \cos x$

$f(a) = \cos\pi = -1$

$f^{(1)}(a) = -\sin\pi = 0$

$f^{(2)}(a) = -\cos\pi = +1$

$f^{(3)}(a) = \sin\pi = 0$

$f^{(4)}(a) = \cos\pi = -1$ (repeats

So:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}$$

Radius = ∞

(Continued from previous page)

(c) $f(x) = \dfrac{x}{9+x^2}$, centered at $a = 0$.

Notice: $\dfrac{x}{9+x^2} = \dfrac{x}{9}\left(\dfrac{1}{1+(x/3)^2}\right) = \dfrac{x}{9}\left(\dfrac{1}{1-\left(-\left(\frac{x}{3}\right)^2\right)}\right)$

so

$$f(x) = \frac{x}{9}\sum_{n=0}^{\infty}\left(-\left(\frac{x}{3}\right)^2\right)^n = \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{3^{2n}}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{3^{2n+2}}$$

The radius of convergence of $\frac{1}{1-x} = \sum x^n$ is $|x|<1$ so, converges for $\left|\left(\frac{x}{3}\right)^2\right| < 1 \Leftrightarrow |x| < 3$. Thus, the radius of convergence is 3.

(d) $f(x) = \ln(5-x)$, centered at $a = 4$.

$$\ln(5-x) = \ln(1+(4-x))$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4-x)^n}{n}$$ (Radius: $|4-x|<$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-4)^n(-1)^n}{n}$$

$$= -\sum_{n=1}^{\infty}\frac{(x-4)^n}{n}$$ Radius $= 1$