ASYMPTOTIC BEHAVIOR AND SYMMETRY OF CONDENSATE SOLUTIONS IN ELECTROWEAK THEORY

ROBIN MING CHEN, YUJIN GUO, AND DANIEL SPIRN

ABSTRACT. We study condensate solutions of a nonlinear elliptic equation in $\mathbb{R}^2$, which models a W-boson with a cosmic string background. The existence and an energy identity of condensate solutions are discussed, based on which the refined asymptotic behavior of condensate solutions is established by studying the corresponding evolution dynamical system. Applying the "shrinking-sphere" method, the symmetry under the inversions of condensate solutions is also proved for some special cases.

1. INTRODUCTION

We consider the semilinear elliptic equation

$$-\Delta u = \lambda e^{au} + |x|^{2N} e^u \tag{1.1}$$

on $\mathbb{R}^2$, where $\lambda > 0$, $a > 0$ and $N \geq 0$ are nonnegative parameters. Equation (1.1) arises from the coupling of electroweak theory to a nonsmooth matter field. In particular, (1.1) is the reduced form of the Bogomoli'nyi-Prasad-Sommerfield type of equation modeling a superconducting cosmic string with a matter field given by the massive W-boson of electroweak theory. We point to Ambjorn-Olesen [1] and Yang [18] for the derivation of (1.1), as well as its physical background from the corresponding Einstein-Weinberg-Salam theory.

In the physical model the coefficient λ of (1.1) is an arbitrary positive constant. However, the parameter N of (1.1) satisfies $N \geq 1$, which denotes

$$N :\equiv \text{ number of cosmic strings.}$$

Therefore, (1.1) can be thought of as the problem of N cosmic strings coinciding at the origin. In view of (10.7.12) in [18], the parameter a of (1.1) satisfies $a = \frac{8\pi G m_W^2}{e^2} > 0$, where G is Newton's gravitational constant, m_W is the mass of the W-boson, and e is the charge of the electron. After considering physical parameters, one finds $a \gg 1$. However, we are mathematically interested in considering (1.1) with general $\lambda > 0$, $a > 0$ and $N \geq 0$.

In this article, we primarily focus on *condensate solutions* $u = u(x)$ of (1.1) in the sense that u satisfies

$$\int_{\mathbb{R}^2} \left[\lambda e^{au(x)} + |x|^{2N} e^{u(x)}\right] dx < \infty, \tag{1.2}$$

and our main purpose is to establish a number of refined qualitative features of such condensate solutions of (1.1). We note that (1.1) has some features in common with elliptic PDEs arising from the prescribed Gaussian curvature problem

$$-\Delta u = R(x) e^u, \tag{1.3}$$

which has been the source of a rich mathematical interest, see [5, 11, 9, 16] and the references therein. Therefore, many of the techniques used here to deduce features of (1.1) originate from the work on (1.3).

As a more general problem of (1.1), one can also consider noncoinciding, multistring configurations satisfying

$$-\Delta u = \lambda e^{au} + \Big(\prod_{i=1}^{N} |x-p_i|^2\Big)e^u \tag{1.4}$$

on $\mathbb{R}^2$. Here the points $\{p_i\}_{i=1}^N$ are given in $\mathbb{R}^2$ allowing multiplicities which correspond to the location on the $(x_1,x_2)-$plane parallel (along the x_3-axis) to the cosmic strings. The problem (1.4) was recently studied by Chae [8], where the author used an implicit function theorem framework to derive the existence of a rich family of condensate solutions u for (1.4) in the sense that u satisfies

$$\int_{\mathbb{R}^2} \Big[\lambda e^{au(x)} + \Big(\prod_{i=1}^{N} |x-p_i|^2\Big)e^{u(x)}\Big]dx < \infty. \tag{1.5}$$

It is clear that Chae's existence is also applied to our problem (1.1). We first provide a simpler method to address the existence of radial condensate solutions of (1.1).

Theorem 1.1. *For any $\alpha \le 0$, (1.1) has a unique radial solution $u = u(|x|)$ satisfying*

$$\begin{cases} -u_{rr} - \dfrac{1}{r}u_r = \lambda e^{au} + r^{2N}e^u, & r = |x| > 0; \\ u(0) = \alpha \ \ \text{and} \ \ u(r) \to -\infty \ \ \text{as} \ \ r \to \infty. \end{cases}$$

Moreover, each radial solution $u = u(r)$ of (1.1) satisfies

$$\max\Big\{\frac{2}{a}, 2(N+1)\Big\} < \int_0^\infty \big[\lambda r e^{au(r)} + r^{2N+1}e^{u(r)}\big]dr < \infty. \tag{1.6}$$

The proof of Theorem 1.1 relies on the shooting argument as employed in [9]. In view of (1.2), one can deduce from (1.6) that any (classical) radial solution of (1.1) must be a condensate solution, and therefore Theorem 1.1 also yields that (1.1) admits infinitely many radial condensate solutions. However, it seems an interesting unsolved question as to whether there exists any symmetry-breaking condensate solution for (1.1), especially in the case where $N > 0$ is sufficiently large.

Our second main result is concerned with asymptotic estimates and an energy identity of condensate solutions of (1.1). By applying Chen and Li's method [11, 12], we show that the energy dictates the asymptotic behavior of condensate solutions.

Theorem 1.2. *Let u be a condensate solution of (1.1) satisfying*

$$\beta := \frac{1}{2\pi}\int_{\mathbb{R}^2} \big[\lambda e^{au(x)} + |x|^{2N}e^{u(x)}\big]dx < \infty, \tag{1.7}$$

then the following asymptotic estimate holds

$$-\beta\ln(|x|+1) - C \le u(x) \le -\beta\ln(|x|+1) + C \qquad \text{in} \quad \mathbb{R}^2 \tag{1.8}$$

with $\beta > \max\big\{2(N+1), \frac{2}{a}\big\}$, and

$$2\int_{\mathbb{R}^2} \Big[N|x|^{2N}e^u + \Big(\frac{1}{a} - 1\Big)\lambda e^{au}\Big]\,dx = \pi\beta(\beta-4). \tag{1.9}$$

The main difficulty of proving Theorem 1.2 is to prove the boundedness of $|x|^{2N}e^{u(x)}$ in $\mathbb{R}^2$, which is addressed in Lemma 3.1. By applying the estimate (1.8), standard elliptic theory implies that any condensate solution of (1.1) satisfying (1.7) must be classical. On the other hand, if $N = 0$ it follows from Theorem 1.2 that any condensate solution of (1.1) admits the asymptotic behavior (1.8) with $\beta > \max\{2, \frac{2}{a}\}$. Following this fact, the symmetry argument of Chen and Li [11] can be directly used to establish that any condensate solution of (1.1) with $N = 0$ must be radially symmetric around some point $\xi \in \mathbb{R}^2$.

The decay rate problem of condensate solutions is of great interest in electroweak theory. Unfortunately, here we are only able to obtain the decay rate $\beta = 4(N + 1)$ for the special case where $a = \frac{1}{N+1}$, which is an immediate result of combining (1.7) and (1.9). For the general case, we guess that the decay rate β is also independent of λ.

Based on the asymptotic estimate (1.8), we next focus on the refined asymptotic behavior of condensate solutions of (1.1) satisfying (1.7) by studying the corresponding evolution dynamical system. Similar to [13] and the references therein, which mostly handle elliptic PDEs with polynomial nonlinearity whereas our problem involves exponential nonlinear terms, our basic tool is to reduce (1.1) into semilinear evolution elliptic problems. Roughly speaking, for any fixed condensate solution u of (1.1), we begin with setting

$$v(t,\theta) := u(r,\theta) + \beta \ln r = u(x) + \beta \ln |x|, \quad t = \ln r \ \text{ and } \ r = |x|, \tag{1.10}$$

where β is as in (1.7) and $(r,\theta) \in (1,\infty) \times S^1$ are polar coordinates in $\mathbb{R}^2 \setminus B_1(0)$, to consider an evolution elliptic equation of the form

$$-v_{tt} - v_{\theta\theta} = \lambda e^{(2-a\beta)t}e^{av} + e^{(2N+2-\beta)t}e^{v}, \quad (t,\theta) \in (0,\infty) \times S^1. \tag{1.11}$$

Since Theorems 1.2 gives the uniform boundedness of v in $(0,\infty) \times S^1$, we shall first prove that there exists a constant $K = K\big(\lambda, a, N, \beta, u(0)\big)$ such that

$$v(t,\theta) = u(x) + \beta \ln |x| \to K \quad \text{as} \quad |x| \to \infty. \tag{1.12}$$

Based on (1.12), we next set a new transformation

$$V(t,\theta) = u(r,\theta) + \beta \ln r - K \quad \text{in} \quad (0,\infty) \times S^1, \tag{1.13}$$

so that $\lim_{t\to+\infty} V(t,\cdot) = 0$, and $V(t,\theta)$ is a uniformly bounded solution of the following evolution elliptic equation

$$-V_{tt} - V_{\theta\theta} = \lambda e^{aK+(2-a\beta)t}e^{aV} + e^{K+(2N+2-\beta)t}e^{V}, \quad (t,\theta) \in (0,\infty) \times S^1. \tag{1.14}$$

By a delicate study of the asymptotic behavior of $V(t,\cdot)$ satisfying (1.14), we finally obtain the following results.

Theorem 1.3. *Let u be a condensate solution of (1.1) satisfying (1.7), and define*

$$\tau := \min\big\{a\beta - 2, \beta - 2(N+1)\big\} > 0.$$

Then there exists a constant $K = K\big(\lambda, a, N, \beta, u(0)\big) \in \mathbb{R}$ such that

$$u(x) \sim -\beta \ln |x| + K \quad \text{as} \quad |x| \to \infty.$$

Moreover, we have the following refined results:

(I). If $\tau \notin \mathbb{N}$, then

(1) *either there exist a natural number k_0 satisfying $1 \le k_0 < \tau$, $A \in \mathbb{R}$ and $\theta_0 \in S^1$ with $A \neq 0$ such that*

$$u(x) \sim -\beta \ln |x| + K + A \sin(k_0\theta + \theta_0)|x|^{-k_0} \quad \text{as} \quad |x| \to \infty, \tag{1.15}$$

(2) *or there exist $A \in \mathbb{R}$ and $\theta_0 \in S^1$ such that*

(a) *If $\beta - 2N > a\beta$, we have the asymptotic behavior*

$$\begin{aligned} u(x) \sim & -\beta \ln|x| + K \\ & + \Big\{ A\sin[(a\beta-2)\theta+\theta_0] - \tfrac{\lambda e^{aK}}{(a\beta-2)^2} \Big\} |x|^{2-a\beta} \quad \textit{as} \quad |x|\to\infty, \end{aligned} \tag{1.16}$$

where $A = 0$ for the case $a\beta - 2 \notin \mathbb{N}$.

(b) *If $\beta - 2N = a\beta$, we have the asymptotic behavior*

$$\begin{aligned} u(x) \sim & -\beta \ln|x| + K \\ & + \Big\{ A\sin[(a\beta-2)\theta+\theta_0] - \tfrac{\lambda e^{aK}+e^K}{(a\beta-2)^2} \Big\} |x|^{2-a\beta} \quad \textit{as} \quad |x|\to\infty, \end{aligned} \tag{1.17}$$

where $A = 0$ for the case $a\beta - 2 \notin \mathbb{N}$.

(c) *If $\beta - 2N < a\beta$, we have the asymptotic behavior*

$$\begin{aligned} u(x) \sim & -\beta \ln|x| + K \\ & + \Big\{ A\sin[(\beta-2N-2)\theta+\theta_0] - \tfrac{e^{K}}{(\beta-2N-2)^2} \Big\} |x|^{2N+2-\beta} \quad \textit{as} \quad |x|\to\infty, \end{aligned} \tag{1.18}$$

where $A = 0$ for the case $\beta - 2N - 2 \notin \mathbb{N}$.

(II). If $\tau \in \mathbb{N}$ and $\tau \geq 2$, then there exist a natural number k_0 satisfying $1 \leq k_0 \leq \tau - 1$, $A \in \mathbb{R}$ and $\theta_0 \in S^1$ such that

$$u(x) \sim -\beta \ln|x| + K + A\sin(k_0\theta + \theta_0)|x|^{-k_0} \quad \textit{as} \quad |x| \to \infty. \tag{1.19}$$

Furthermore, if (1.19) holds for k_0 satisfying $1 \leq k_0 \leq \tau - 2$, then it necessarily has $A \neq 0$ in (1.19).

When $a = \frac{1}{N+1}$, Theorem 1.2 gives that $\beta = 4(N+1)$ and hence $\tau = 2$. However, we note from [11, 16] that when $\lambda = 0$, then $\beta = 4(N+1)$ and hence $\beta - 2N - 2 = 2(N+1) \geq 2$ can be either an integer or not, depending on the value of $N \geq 0$. Therefore, we guess that $\tau \geq 2$ holds for any $a > 0$ and $N \geq 0$, and τ can be either an integer or not, depending on the values of $a > 0$ and $N \geq 0$. We also note that when $N = 0$ and $a = 1$, then $\beta = 4$ and hence (1.15) matches well with the asymptotic behavior of exact solutions of (1.1) discussed in [11, 16].

If $u = u(r)$ is a radial condensate solution of (1.1) satisfying (1.7), then in Section 4 we shall obtain the following more precise asymptotic behavior.

Proposition 1.4. *Let $u = u(r)$ be a radial condensate solution of (1.1) satisfying (1.7), then there exists a constant $K = K\big(\lambda, a, N, \beta, u(0)\big) \in \mathbb{R}$ such that*

(1) *If $\beta - 2N > a\beta$, we have the asymptotic behavior*

$$u(r) \sim -\beta \ln r + K - \frac{\lambda e^{aK}}{(a\beta-2)^2} r^{2-a\beta} \quad \textit{as} \quad r = |x| \to \infty. \tag{1.20}$$

(2) *If $\beta - 2N = a\beta$, we have the asymptotic behavior*

$$u(r) \sim -\beta \ln r + K - \frac{\lambda e^{aK} + e^K}{(a\beta-2)^2} r^{2-a\beta} \quad \textit{as} \quad r = |x| \to \infty. \tag{1.21}$$

(3) *If $\beta - 2N < a\beta$, we have the asymptotic behavior*

$$u(r) \sim -\beta \ln r + K - \frac{e^{K}}{(\beta-2N-2)^2} r^{2N+2-\beta} \quad \textit{as} \quad r = |x| \to \infty. \tag{1.22}$$

The results of Theorem 1.3 and Proposition 1.4 go only to the case where $\lambda > 0$. However, our methods can also be applied to (1.1) with $\lambda = 0$, in which case some similar results can be obtained, see Remark 4.3 for more discussions. Also, we expect that our methods may be applicable to some other types of elliptic problems.

By applying the "shrinking-sphere" method, c.f. [16], our fourth main result is devoted to the following "symmetry under the inversions", $i.e.$, the invariance of (1.1) with respect to "inversions", for the special case $a = \frac{1}{N+1}$.

Theorem 1.5. *Let $a = \frac{1}{N+1}$, and suppose $u(x)$ is a condensate solution of (1.1) satisfying (1.7), and let $K = K\big(\lambda, a, N, \beta, u(0)\big)$ be the constant given in Theorem 1.3. Then*

$$u(x) = u\Big(\frac{x}{\tau|x|^2}\Big) + 2(N+1)\ln\frac{1}{\tau|x|^2} \quad \text{in} \quad \mathbb{R}^2, \tag{1.23}$$

where

$$\tau = e^{(u(0)-K)/2(N+1)}. \tag{1.24}$$

Moreover, u satisfies

$$\begin{cases} (r - \frac{1}{\sqrt{\tau}})(r\partial_r u + 2N + 2) < 0 & \text{if} \quad r = |x| \neq \frac{1}{\sqrt{\tau}}, \\ r\partial_r u + 2(N+1) = 0 & \text{if} \quad r = |x| = \frac{1}{\sqrt{\tau}}. \end{cases} \tag{1.25}$$

One can note that Theorem 1.5 illustrates the dependence on $u(0)$ of the constant $K = K\big(\lambda, a, N, \beta, u(0)\big)$ given in Theorem 1.3.

As addressed in Section 6, the last purpose of this article is to study the asymptotic behavior of condensate solutions of the noncoinciding problem (1.4). Similar to Theorem 1.2, we shall establish the following results.

Theorem 1.6. *Let u be a condensate solution of (1.4) satisfying*

$$\beta := \frac{1}{2\pi}\int_{\mathbb{R}^2}\big[\lambda e^{au(x)} + \big(\prod_{i=1}^{N}|x-p_i|^2\big)e^{u(x)}\big]dx < \infty, \tag{1.26}$$

then the following asymptotic estimate holds

$$-\beta\ln(|x|+1) - C \le u(x) \le -\beta\ln(|x|+1) + C \qquad \text{in} \quad \mathbb{R}^2 \tag{1.27}$$

with $\beta > \max\big\{2(N+1), \frac{2}{a}\big\}$, and

$$\int_{\mathbb{R}^2}\Big[\big(x\cdot\nabla(\prod_{i=1}^{N}|x-p_i|^2)\big)e^u + \Big(\frac{2}{a} - 2\Big)\lambda e^{au}\Big]\,dx = \pi\beta(\beta - 4). \tag{1.28}$$

As discussed in Section 6, by following Theorem 1.6 one can apply the argument of Section 4 to deriving further the refined asymptotic behavior of condensate solutions for (1.4).

This article is organized as follows. In Section 2 we discuss the existence and energy estimates of radial solutions for (1.1) by proving Theorem 1.1, and we show that any (classical) radial solutions of (1.1) must be a condensate solution. In Section 3 we prove Theorem 1.2 on the energy identity and asymptotic estimates of general condensate solutions of (1.1). Theorem 1.3 and Proposition 1.4 are then proved in Section 4, which give the refined asymptotic behavior of condensate solutions of (1.1). In Section 5, we prove Theorem 1.5 on the "symmetry under the inversions" of condensate solutions of (1.1). Finally, we sketch the proof of Theorem 1.6 in Section 6.

2. ON THE RADIAL CONDENSATE SOLUTIONS

This section is devoted to the proof of Theorem 1.1 on the existence and energy estimates of radially symmetric condensate solutions of (1.1).

We first address the existence of radially symmetric solutions by the shooting argument as used in [9]. Towards this purpose, we consider the radial solution $u = u(r)$ of (1.1) satisfying

$$-u_{rr} - \frac{1}{r}u_r = \lambda e^{au} + r^{2N}e^u, \quad r = |x| > 0, \tag{2.1}$$

$$u_r(0) = 0 \text{ and } u(0) = \alpha, \tag{2.2}$$

where $\alpha \in \mathbb{R}$ is a parameter. By changing the variable $t = \ln r$, then $u = u(t)$ satisfies

$$-u'' = \lambda e^{2t}e^{au} + e^{2(N+1)t}e^u, \quad -\infty < t < +\infty, \tag{2.3}$$

$$u(t) \to \alpha \quad \text{as } t \to -\infty. \tag{2.4}$$

We truncate (2.3) into the form

$$-u'' = \lambda e^{2t}f(u) + e^{2(N+1)t}g(u), \quad -\infty < t < +\infty, \tag{2.5}$$

where $f(u) = e^{au}$ and $g(u) = e^u$ for $u \le 0$, and both $f(u) \ge 0$ and $g(u) \ge 0$ are smoothly defined for $u > 0$ in such a way that

$$\beta_1 = \sup_{u\in\mathbb{R}} \Big\{1 + |f(u)| + |g(u)| + |f'(u)| + |g'(u)|\Big\} < \infty\,.$$

We start with the following existence and uniqueness.

Lemma 2.1. *For each $\alpha \in \mathbb{R}$, the problem (2.4)–(2.5) admits a unique solution $u(t)$ satisfying*

$$u(t) = \alpha + o(1) \quad \textit{as} \quad t \to -\infty. \tag{2.6}$$

Proof. It is easy to verify that $u = u(t)$ is a solution of (2.4)–(2.5), if and only if $u = u(t)$ satisfies

$$u(t) = \alpha - \int_{-\infty}^{t} (t-s)\Big[\lambda e^{2s}f\big(u(s)\big) + e^{2(N+1)s}g\big(u(s)\big)\Big]ds, \tag{2.7}$$

and hence (2.6) is a direct result of (2.7). Let $T \in \mathbb{R}$ be a constant satisfying

$$\lambda e^{2T} + \frac{e^{2(N+1)T}}{(N+1)^2} \le \frac{2}{\beta_1}. \tag{2.8}$$

By constructing a Picard iterative sequence $\{u_i\}$: $u_0(t) \equiv \alpha$,

$$u_i(t) = \alpha - \int_{-\infty}^{t} (t-s)\Big[\lambda e^{2s}f\big(u_{i-1}(s)\big) + e^{2(N+1)s}g\big(u_{i-1}(s)\big)\Big]ds, \quad i \ge 1,$$

it yields that for any $t \in (-\infty, T]$,

$$\begin{aligned}
&\sup\nolimits_{t\in(-\infty,T]} |u_{i+1}(t) - u_i(t)| \\
\le\ & \frac{1}{4}\Big[\lambda e^{2T} \sup_{u\in\mathbb{R}} |f'(u)| + \frac{e^{2(N+1)T}}{(N+1)^2} \sup_{u\in\mathbb{R}} |g'(u)|\Big] \sup_{t\in(-\infty,T]} |u_i(t) - u_{i-1}(t)| \\
\le\ & \frac{1}{2} \sup_{t\in(-\infty,T]} |u_i(t) - u_{i-1}(t)|.
\end{aligned}$$

This shows that there exists a solution of (2.7) in the interval $(-\infty, T]$. Since $f(u)$ and $g(u)$ are bounded, the solution can be extended into $\mathbb{R}$, which then gives the existence of solutions for (2.4)–(2.5).

We now prove the uniqueness of solutions for (2.4)–(2.5). Assume that U_1 and U_2 are two solutions of (2.7) in the interval $(-\infty, T]$, where T satisfies (2.8). Then the difference $U = U_1 - U_2$ satisfies

$$\begin{aligned}
&\sup_{t\in(-\infty,T]} |U(t)| \\
&= \sup_{t\in(-\infty,T]} \Big| \int_{-\infty}^{t} (t-s)\Big\{\lambda e^{2s}\big[f(U_1)-f(U_2)\big] + e^{2(N+1)s}\big[g(U_1)-g(U_2)\big]\Big\}ds\Big| \\
&\le \frac{\beta_1}{4}\Big(\lambda e^{2T} + \frac{e^{2(N+1)T}}{(N+1)^2}\Big) \sup_{t\in(-\infty,T]} |U(t)| \le \frac{1}{2} \sup_{t\in(-\infty,T]} |U(t)|.
\end{aligned} \tag{2.9}$$

This inequality implies that $\sup_{t\in(-\infty,T]} |U(t)| = 0$, which then gives $U_1 \equiv U_2$ in $(-\infty, T]$. Since $f(u)$ and $g(u)$ are bounded, the uniqueness of solutions for (2.4)–(2.5) in $\mathbb{R}$ then easily follows from the unique continuation of solutions. □

We next prove the existence of non-positive solutions for (2.4)–(2.5), which then implies the existence of (non-positive) radial solutions for (1.1). For this purpose, we vary the parameter $\alpha \in \mathbb{R}$, and denote by $u(t,\alpha)$ the solution obtained by Lemma 2.1. For convenience, we set $u'(t,\alpha) = \frac{\partial}{\partial t}u(t,\alpha)$ and $u_\alpha(t,\alpha) = \frac{\partial}{\partial \alpha}u(t,\alpha)$.

Define

$$\begin{aligned}
A^+ &= \big\{\alpha \in \mathbb{R} : \ u(t_+,\alpha) > 0 \text{ for some } \ t_+ \in \mathbb{R}\big\}, \\
A^0 &= \big\{\alpha \in \mathbb{R} : \ u(t,\alpha) \le 0 \ \text{ and } u'(t,\alpha) \ge 0 \text{ for any } \ t \in \mathbb{R}\big\}, \\
A^- &= \big\{\alpha \in \mathbb{R} : \ u(t,\alpha) \le 0 \text{ for any } \ t \in \mathbb{R}, \text{ and } u'(t_-,\alpha) < 0 \text{ for some } \ t_- \in \mathbb{R}\big\}.
\end{aligned}$$

It is clear that

$$A^+ \cup A^0 \cup A^- = \mathbb{R}, \quad A^+ \cap A^0 = A^+ \cap A^- = A^- \cap A^0 = \varnothing.$$

Lemma 2.2. *The following statements hold:*

(1) $A^0 = \varnothing$;
(2) A^+ *is a non-empty open set, and if* $\alpha \in A^+$ *then* $u' > 0$ *in the set* $\{t : u(\tau,\alpha) < 0 \ holds \ for \ any \ \tau < t\}$.
(3) $A^- = \mathbb{R} \setminus A^+$, *and if* $\alpha \in A^-$ *then* $u'' < 0$, $u < 0$ *in* $\mathbb{R}$, *and* $\lim_{t\to\infty} u(t,\alpha) = -\infty$.

Proof. 1. Since the equation (2.5) implies $u'' \le 0$ in $\mathbb{R}$, if $\alpha \in A^0$ then we have $u' \ge 0$ in $\mathbb{R}$ and $b = \lim_{t\to\infty} u(t,\alpha) \le 0$ exists. Furthermore, it follows from (2.5) that

$$\begin{aligned}
\lim_{t\to\infty} u''(t,\alpha) &= -\lim_{t\to\infty} \Big[\lambda e^{2t} f\big(u(t,\alpha)\big) + e^{2(N+1)t} g\big(u(t,\alpha)\big)\Big] \\
&= -\lambda e^{ab} \lim_{t\to\infty} e^{2t} - e^{b} \lim_{t\to\infty} e^{2(N+1)t} = -\infty,
\end{aligned}$$

which is impossible by the definition of A^0. Therefore, $A^0 = \varnothing$.

2. Since $f(u)$ and $g(u)$ are bounded and nonnegative, we first note from (2.7) that $u(0,\alpha) > 0$ holds for sufficiently large α. This shows that A^+ is non-empty.

Similar to the uniqueness proof of Lemma 2.1, one can deduce that $u(t,\alpha)$ is continuous in α. Therefore, if $u(t_+,\alpha_+) > 0$ then $u(t_+,\alpha) > 0$ for α sufficiently close to α_+, which proves that A^+ is an open set.

Furthermore, if $\alpha \in A^+$ we let t_+ be the first time at which $u(t,\alpha)$ hits the $t-$axis from below. Then (2.7) implies that $u(t,\alpha) < 0$ and $u''(t,\alpha) < 0$ in $(-\infty, t_+)$. This implies $u'(t,\alpha) > 0$ in $(-\infty, t_+)$.

3. It now concludes from above proofs that $A^- = \mathbb{R} \setminus A^+$. Moreover, if $\alpha \in A^-$ then (2.5) implies that $u''(t,\alpha) < 0$ in $\mathbb{R}$, and (2.7) implies that $u(t,\alpha) < 0$ in $\mathbb{R}$.

Hence, $u'(t,\alpha)$ strictly decreases in t, which implies that $\lim_{t\to\infty} u'(t,\alpha) < 0$. This gives $\lim_{t\to\infty} u(t,\alpha) = -\infty$. □

It is now easy to deduce from (2.7) that $A^+ = (0,+\infty)$ and $A^- = (-\infty, 0]$, which gives the existence and infinite multiplicity of radial solutions for (1.1). This proves the first part of Theorem 1.1. The second part of Theorem 1.1 is a direct consequence of the following lemma, which also implies that any (classical) radial solution of (1.1) must be a condensate solution.

Lemma 2.3. *Suppose $u = u(r)$ is a classical radial solution of (1.1), then we have the estimates*

$$M := \max\Big\{\frac{2}{a}, 2(N+1)\Big\} < \gamma := -\lim_{r\to\infty} r u_r(r) = \int_0^\infty \big[\lambda s e^{au(s)} + s^{2N+1} e^{u(s)}\big] ds < \infty. \tag{2.10}$$

Proof. For any classical radial solution $u = u(r)$ of (1.1), we first note that

$$r u_r = -\int_0^r \big[\lambda s e^{au(s)} + s^{2N+1} e^{u(s)}\big] ds, \quad r \ge 0. \tag{2.11}$$

Since $(r u_r)_r < 0$ for any $r > 0$, it follows that $\gamma := -\lim_{r\to\infty} r u_r(r)$ exists and satisfies $0 < \gamma \le \infty$.

We first prove the lower estimate of γ. Suppose that $0 < \gamma \le M := \max\big\{\frac{2}{a}, 2(N+1)\big\}$. Since $-r u_r(r)$ is strictly increasing, we have

$$-r u_r \le \gamma \le M := \max\Big\{\frac{2}{a}, 2(N+1)\Big\}, \quad \text{i.e.} \quad u(r) \ge u(1) - M \ln r \quad \text{for} \quad r \ge 1.$$

This implies that

$$\begin{aligned} -r u_r(r) &= -u_r(1) + \int_1^r \big[\lambda s e^{au(s)} + s^{2N+1} e^{u(s)}\big] ds \\ &\ge -u_r(1) + \int_1^r \big[\lambda e^{au(1)} s^{1-aM} + e^{u(1)} s^{2N+1-M}\big] ds, \end{aligned}$$

which leads to

$$\gamma \ge -u_r(1) + \int_1^\infty \big[\lambda e^{au(1)} s^{1-aM} + e^{u(1)} s^{2N+1-M}\big] ds = \infty,$$

a contradiction. Therefore, we have the lower estimate $\gamma > M := \max\big\{\frac{2}{a}, 2(N+1)\big\}$.

We next prove the finiteness of γ. Since $\gamma > M := \max\big\{\frac{2}{a}, 2(N+1)\big\}$, it follows that for any small $\epsilon > 0$ such that

$$2\epsilon < \gamma - M, \text{ i.e. } \gamma - \epsilon > M + \epsilon, \tag{2.12}$$

there exists a fixed $R = R(\epsilon) > 1$ with

$$u(r) \le u(R) - (\gamma - \epsilon) \ln \frac{r}{R}, \quad r \ge R.$$

This leads to

$$\begin{aligned} -r u_r(r) &= -R u_r(R) + \int_R^r \big[\lambda s e^{au(s)} + s^{2N+1} e^{u(s)}\big] ds \\ &\le -R u_r(R) + \int_R^r \Big[\lambda R e^{au(R)} \big(\frac{s}{R}\big)^{1-(\gamma-\epsilon)a} + R^{2N+1} e^{u(R)} \big(\frac{s}{R}\big)^{2N+1-(\gamma-\epsilon)}\Big] ds. \end{aligned}$$

Therefore, we then reduce from (2.12) that

$$\begin{aligned}\gamma &\le -Ru_r(R) + \int_R^\infty \big[\lambda R e^{au(R)}\big(\frac{s}{R}\big)^{1-(\gamma-\epsilon)a} + R^{2N+1}e^{u(R)}\big(\frac{s}{R}\big)^{2N+1-(\gamma-\epsilon)}\big]ds\\ &\le C_1 + \int_R^\infty \big[\lambda R e^{au(R)}\big(\frac{s}{R}\big)^{1-(M+\epsilon)a} + R^{2N+1}e^{u(R)}\big(\frac{s}{R}\big)^{2N+1-(M+\epsilon)}\big]ds\\ &\le C_1 + C_2 < \infty,\end{aligned}$$

since $M+\epsilon > \max\big\{\frac{2}{a}, 2(N+1)\big\}$, and hence the lemma follows. □

Remark 2.1. Suppose $u = u(r)$ is a classical radial solution of (1.1), and let γ be defined as in (2.10). It then follows from Lemma 2.3 that $u = u(r)$ must be a condensate solution of (1.1). Moreover, one can deduce from (1.1) that $u = u(r)$ satisfies the identity

$$\big[\frac{1}{2}\big(ru_r(r)\big)^2 + \frac{\lambda}{a}r^2 e^{au(r)} + r^{2(N+1)}e^{u(r)}\big]_r = \frac{2\lambda}{a}re^{au(r)} + 2(N+1)r^{2N+1}e^{u(r)}. \quad (2.13)$$

For any $R > 0$, integrating both hand sides of (2.13) over $[0, R]$ now yields that

$$\int_0^R \big[\frac{2\lambda}{a}re^{au(r)} + 2(N+1)r^{2N+1}e^{u(r)}\big]dr = \big[\frac{1}{2}\big(ru_r(r)\big)^2 + \frac{\lambda}{a}r^2e^{au(r)} + r^{2(N+1)}e^{u(r)}\big]\Big|_{r=R}. \quad (2.14)$$

Since Lemma 2.3 implies that

$$\frac{\lambda}{a}r^2e^{au(r)} + r^{2(N+1)}e^{u(r)} \to 0 \quad \text{as} \quad r \to \infty,$$

it now follows from (2.14), by setting $R = \infty$, that

$$\frac{1}{2}\gamma^2 = \int_0^\infty \big[\frac{2\lambda}{a}re^{au(r)} + 2(N+1)r^{2N+1}e^{u(r)}\big]dr, \quad (2.15)$$

which is equivalent to

$$\gamma(\gamma-4) = 4\int_0^\infty \big[Nr^{2N}e^{u(r)} + \big(\frac{1}{a}-1\big)\lambda e^{au(r)}\big]rdr. \quad (2.16)$$

This shows that the energy identity (1.9) can be easily established in the class of radial condensate solutions.

3. Asymptotic Estimates and Energy Identity

The main purpose of this section is to prove Theorem 1.2 on the asymptotic estimates and energy identity of general condensate solutions of (1.1). Motivated by Brezis and Merle's estimate, cf. Theorem 2 in [5], we begin with the following crucial lemma, and our main difficulty lies in how to prove the boundedness of $|x|^{2N}e^{u(x)}$ in $\mathbb{R}^2$.

Lemma 3.1. *Suppose u is a solution of (1.1) satisfying*

$$\beta := \frac{1}{2\pi}\int_{\mathbb{R}^2}\big[\lambda e^{au(x)} + |x|^{2N}e^{u(x)}\big]dx < \infty. \quad (3.1)$$

Then e^{au} and $|x|^{2N}e^u$ are bounded in $\mathbb{R}^2$.

Proof. We first prove that e^{au} and $|x|^{2N}e^u$ are bounded in the exterior domain $B_R^c(0) := \{x \in \mathbb{R}^2 : |x| \ge R\} \subset\subset \mathbb{R}^2$, where $1 < R < \infty$ is fixed.

Consider any fixed point $x_0 \in \mathbb{R}^2$ satisfying $|x_0| \ge R+1$, and let $B_1(x_0)$ be a unit ball centered at x_0. Set

$$f(x) = u(x) + 2N\ln|x_0|, \quad (3.2)$$

such that $f = f(x)$ satisfies the equation

$$-\Delta f = \big(\frac{|x|}{|x_0|}\big)^{2N} e^f + \frac{\lambda}{|x_0|^{2Na}} e^{af} \quad \text{in } B_R^c(0). \tag{3.3}$$

Note that for above choices of R and x_0, the assumption (3.1) implies

$$\int_{B_R^c(0)} \Big[\big(\frac{|x|}{|x_0|}\big)^{2N} e^f + \frac{\lambda}{|x_0|^{2Na}} e^{af}\Big] dx < \infty.$$

This gives that for any $0 < \epsilon < \min\left\{1, \frac{1}{a}\right\}$, one can split

$$\big(\frac{|x|}{|x_0|}\big)^{2N} e^f + \frac{\lambda}{|x_0|^{2Na}} e^{af} = P_1 + P_2 \quad \text{in } B_R^c(0),$$

such that $\|P_1\|_{L^1(B_R^c(0))} < \epsilon$ and $P_2 \in L^\infty(B_R^c(0))$. Let f_i, $i = 1, 2$, be the solution of

$$\begin{cases} -\Delta f_i = P_i & \text{in} \quad B_1(x_0), \\ \quad f_i = 0 & \text{on} \quad \partial B_1(x_0). \end{cases}$$

Denote by C various constants independent of x_0, but possibly depending on ϵ. It then follows from Theorem 1 in [5] that

$$\int_{B_1(x_0)} e^{\frac{|f_1|}{\epsilon}} dx \le C,$$

which particularly gives that $\|f_1\|_{L^1(B_1(x_0))} \le C$. Note that the standard regularity implies $\|f_2\|_{L^\infty(B_1(x_0))} \le C$.

Let $f_3 = f - f_1 - f_2$ so that $\Delta f_3 = 0$ in $B_1(x_0)$, then the mean value theorem yields

$$\|f_3^+\|_{L^\infty(B_{1/2}(x_0))} \le C\|f_3^+\|_{L^1(B_1(x_0))}. \tag{3.4}$$

Since

$$a\int_{B_{1/2}(x_0)} f^+ \, dx \le \int_{B_{1/2}(x_0)} e^{af} \, dx \le C,$$

by applying $f_3^+ \le f^+ + |f_1| + |f_2|$, we now obtain from (3.4) that $\|f_3\|_{L^\infty(B_{1/2}(x_0))} \le C$. Therefore, we obtain that

$$\|e^{f_1}\|_{L^{\frac{1}{\epsilon}}(B_1(x_0))} \le C, \quad \|f_2\|_{L^\infty(B_1(x_0))} \le C, \quad \|f_3\|_{L^\infty(B_{1/2}(x_0))} \le C, \tag{3.5}$$

together with $\|f_1\|_{L^1(B_1(x_0))} \le C$.

We now rewrite (3.3) as

$$\begin{aligned} -\Delta f &= \big(\frac{|x|}{|x_0|}\big)^{2N} e^f + \frac{\lambda}{|x_0|^{2Na}} e^{af} \\ &= \Big(\big(\frac{|x|}{|x_0|}\big)^{2N} e^{f_1}\Big) e^{f_2+f_3} + \Big(\frac{\lambda}{|x_0|^{2Na}} e^{af_1}\Big) e^{a(f_2+f_3)} := g \quad \text{in } B_1(x_0). \end{aligned} \tag{3.6}$$

For any $\lambda > 0$ and $N \ge 0$, we next prove that $\|g\|_{L^{1+\delta}(B_{1/2}(x_0))} \le C$ holds for some $\delta > 0$, by separately considering the following two different cases of a.

Case 1: $a \ge 1$**.** In this case, since

$$e^{a(f_2+f_3)}, \; e^{(f_2+f_3)} \in L^\infty(B_{1/2}(x_0)), \quad e^{f_1} \in L^{1/\epsilon}(B_1(x_0)),$$

where $0 < \epsilon < 1/a \leq 1$, it follows from (3.6) that

$$\begin{aligned}\int_{B_{1/2}(x_0)} g^{\frac{1}{a\epsilon}}dx &< C\int_{B_{1/2}(x_0)}\Big(\big(\frac{|x|}{|x_0|}\big)^{2N}e^{f_1}+\frac{\lambda}{|x_0|^{2Na}}e^{af_1}\Big)^{\frac{1}{a\epsilon}}dx\\ &< C\int_{B_{1/2}(x_0)} e^{\frac{f_1}{a\epsilon}}dx + C\int_{B_{1/2}(x_0)} e^{\frac{f_1}{\epsilon}}dx\\ &< C\big(\frac{\pi}{4}\big)^{\frac{a-1}{a}}\Big[\int_{B_{1/2}(x_0)} e^{\frac{f_1}{\epsilon}}dx\Big]^{\frac{1}{a}} + C_2\\ &< C_1 + C_2 < \infty\end{aligned}$$

uniformly as $x_0 \to \infty$, where the bounded constants C_1 and C_2 are independent of x_0. If $a > 1$, here we apply Hölder inequality in the third inequality. Therefore, we conclude from $0 < \epsilon < 1/a \leq 1$ that $\|g\|_{L^{1+\delta}(B_{1/2}(x_0))} \leq C$ holds for some $\delta > 0$.

Case 2: $0 < a < 1$**.** In this case, since $0 < \epsilon < 1$, it then follows from (3.5) and (3.6) that

$$\begin{aligned}\int_{B_{1/2}(x_0)} g^{\frac{1}{\epsilon}}dx &< C\int_{B_{1/2}(x_0)}\Big(\big(\frac{|x|}{|x_0|}\big)^{2N}e^{f_1}+\frac{\lambda}{|x_0|^{2Na}}e^{af_1}\Big)^{\frac{1}{\epsilon}}dx\\ &< C\int_{B_{1/2}(x_0)} e^{\frac{f_1}{\epsilon}}dx + C\int_{B_{1/2}(x_0)} e^{\frac{af_1}{\epsilon}}dx\\ &< C_3 + C\big(\frac{\pi}{4}\big)^{1-a}\Big[\int_{B_{1/2}(x_0)} e^{\frac{f_1}{\epsilon}}dx\Big]^{a}\\ &< C_3 + C_4 < \infty\end{aligned}$$

uniformly as $x_0 \to \infty$, where the bounded constants C_3 and C_4 are independent of x_0. Here we again apply Hölder inequality in the third inequality. Therefore, in the case $0 < a < 1$, we also have $\|g\|_{L^{1+\delta}(B_{1/2}(x_0))} \leq C$ for some $\delta > 0$.

We now obtain from the mean value theorem and standard elliptic estimates that

$$\|f^+\|_{L^\infty(B_{1/4}(x_0))} \leq C\|f^+\|_{L^1(B_{1/2}(x_0))} + C\|g\|_{L^{1+\delta}(B_{1/2}(x_0))} \leq C, \tag{3.7}$$

where C is independent of x_0. Since x_0, with $|x_0| \geq R+1$, is arbitrary, we conclude from this and (3.2) that

$$u(x) \leq C - 2N\ln(|x|+1) \quad \text{in} \quad B_R^c(0) := \big\{x \in \mathbb{R}^2 : |x| \geq R\big\},$$

which implies that e^{au} and $|x|^{2N}e^u$ are bounded in $B_R^c(0)$ for any fixed $1 < R < \infty$.

For above fixed $0 < R < \infty$, we finally consider the condensate solution $u(x)$ of (1.1) in $B_{2R}(0)$, instead of considering the solution of (3.3) in $B_R^c(0)$. A similar argument as above then yields that $u(x)$ is bounded in $B_{2R}(0)$. Therefore, e^{au} and $|x|^{2N}e^u$ are bounded in $\mathbb{R}^2$. □

Lemma 3.2. *If u is a solution of (1.1) satisfying (3.1), then we have $\beta > \max\{2(N+1), 2/a\}$, and*

$$-\beta\ln(|x|+1) - C \leq u(x) \leq -\beta\ln(|x|+1) + C \quad \textit{in} \quad \mathbb{R}^2 \tag{3.8}$$

for some positive constant C.

Proof. Based on Lemma 3.1, the proof of Lemma 3.2 is similar to that of Lemma 1.1 in [11]. Here we give the details for completeness.

(i). Define

$$w(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\Big(\ln|x-y| - \ln(|y|+1)\Big)\Big(\lambda e^{au(y)} + |y|^{2N}e^{u(y)}\Big)dy. \tag{3.9}$$

We claim that

$$\frac{w(x)}{\ln|x|} \to \beta \quad \text{uniformly as} \quad |x| \to \infty. \tag{3.10}$$

Indeed, since Lemma 3.1 gives that $\lambda e^{au(x)}$ and $|x|^{2N}e^{u(x)}$ are bounded in $\mathbb{R}^2$, we first note that $w(x)$ is well-defined and satisfies

$$\Delta w(x) = \lambda e^{au(x)} + |x|^{2N}e^{u(x)}, \quad x \in \mathbb{R}^2.$$

Set

$$I = \int_{\mathbb{R}^2} \frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|}\Big(\lambda e^{au(y)} + |y|^{2N}e^{u(y)}\Big)dy \tag{3.11}$$

such that $I = 2\pi\big(\frac{w(x)}{\ln|x|} - \beta\big)$. Write $I = I_1 + I_2 + I_3$, where I_1, I_2 and I_3 are the integrals on the regions $D_1 = \{y : \ |x-y| \le 1\}$, $D_2 = \{y : \ |x-y| > 1, |y| \le K\}$ and $D_3 = \{y : \ |x-y| > 1, |y| > K\}$, respectively, where the constant $K > 0$ is to be determined later. Here we may assume that $|x| \ge 3$, since we consider $|x| \to \infty$. Firstly, to estimate I_1, we have

$$\begin{aligned} I_1 \le C &\int_{D_1} \Big(\lambda e^{au(y)} + |y|^{2N}e^{u(y)}\Big)dy \\ &+ \frac{1}{\ln|x|}\int_{D_1} \ln|x-y|\Big(\lambda e^{au(y)} + |y|^{2N}e^{u(y)}\Big)dy. \end{aligned}$$

Then by the boundedness of $\lambda e^{au(x)}$ and $|x|^{2N}e^{u(x)}$ in $\mathbb{R}^2$, together with the assumption (3.1), it easily follows that $I_1 \to 0$ as $|x| \to \infty$. Secondly, for each fixed K, in D_2 we have

$$\frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|} \to 0 \quad \text{as} \quad |x| \to \infty,$$

and hence $I_2 \to 0$ as $|x| \to \infty$. Thirdly, if $|x-y| > 1$ and $|x| \ge 3$, we then have

$$\Big|\frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|}\Big| \le C,$$

and hence by choosing $K \to \infty$ we obtain that $I_3 \to 0$ as $|x| \to \infty$. Therefore, we obtain from (3.11) that

$$I \to 0 \ \text{ as } \ |x| \to \infty,$$

which then yields the claim (3.10).

(ii). We now consider the function $v(x) = u(x) + w(x)$. Then $\Delta v(x) = 0$ and since

$$\ln|x-y| - \ln(|y|+1) \le \ln(|x|+1),$$

we have

$$w(x) \le \frac{1}{2\pi}\ln(|x|+1)\int_{\mathbb{R}^2} \Big(\lambda e^{au(y)} + |y|^{2N}e^{u(y)}\Big)dy = \beta\ln(|x|+1).$$

This leads to

$$v(x) \le C + \beta\ln(|x|+1)$$

for some constant C. Hence v must be a constant, and it therefore reduces from (3.10) that

$$\frac{u(x)}{\ln|x|} \to -\beta \quad \text{uniformly as} \quad |x| \to \infty. \tag{3.12}$$

(iii). By the above proof, we can write the solution of (1.1) as

$$u(x) = C - \frac{1}{2\pi}\int_{\mathbb{R}^2} \Big(\ln|x-y| - \ln(|y|+1)\Big)\Big(\lambda e^{au(y)} + |y|^{2N}e^{u(y)}\Big)dy. \tag{3.13}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^2}\Big(\ln|x-y|-\ln(|y|+1)\Big)\Big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\Big)dy\\&\le \ln(|x|+1)\int_{\mathbb{R}^2}\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy\\&=2\pi\beta\ln(|x|+1),\end{aligned}$$

it then follows from (3.13) that

$$u(x)\ge -\beta\ln(|x|+1)-C.$$

This gives the left inequality of (1.8), and the lower estimate $\beta>\max\{2(N+1),2/a\}$ follows from the assumption (3.1).

(iv). To prove the right inequality of (1.8), it now suffices to prove that

$$w(x)\ge \beta\ln(|x|+1)-C \quad \text{in} \quad \mathbb{R}^2. \tag{3.14}$$

In fact, we have for $|x-y|\ge 1$,

$$|x|\le |x-y|(|y|+1),$$

which implies that

$$\ln|x|-2\ln(|y|+1)\le \ln|x-y|-\ln(|y|+1).$$

Consequently,

$$\begin{aligned}w(x)\ge{}&\frac{1}{2\pi}\int_{|x-y|\ge1}\big(\ln|x|-2\ln(|y|+1)\big)\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy\\&+\frac{1}{2\pi}\int_{|x-y|\le1}\big(\ln|x-y|-2\ln(|y|+1)\big)\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy\\\ge{}&\beta\ln|x|-\frac{\ln|x|}{2\pi}\int_{|x-y|\le1}\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy\\&+\frac{1}{2\pi}\int_{|x-y|\le1}\ln|x-y|\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy\\&-\frac{1}{\pi}\int_{\mathbb{R}^2}\ln(|y|+1)\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy\\={}&\beta\ln|x|+I_1+I_2+I_3.\end{aligned}$$

In view of (3.12), where $\beta>\max\{2/a,2(N+1)\}$, one can observe that I_1, $I_2\to 0$ as $|x|\to\infty$ and I_3 is finite. Since $\lambda e^{au(x)}$ and $|x|^{2N}e^{u(x)}$ are bounded in $\mathbb{R}^2$, (3.14) now follows. □

Lemma 3.2 shows that for any $a>0$, if $u(x)$ is a condensate solution of (1.1), then u decays very fast with the rate $\beta>\max\{2(N+1),2/a\}>2$. Based on this fact, the symmetry argument of Chen and Li [11] can be directly used to establish that any condensate solution of (1.1) with $N=0$ must be radially symmetric around some point $\xi\in\mathbb{R}^2$.

Let (r,θ) be the polar coordinates in $\mathbb{R}^2$, it then follows from (3.13) that

$$\begin{aligned}ru_r&=x_1u_{x_1}+x_2u_{x_2}\\&=-\beta-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{y\cdot(x-y)}{|x-y|^2}\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy,\end{aligned}$$

and

$$u_\theta=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\bar{y}\cdot(x-y)}{|x-y|^2}\big(\lambda e^{au(y)}+|y|^{2N}e^{u(y)}\big)dy,$$

where $\bar{y} = (y_2, -y_1) \in \mathbb{R}^2$. Following this notation, one can further establish the following estimates.

Lemma 3.3. *If u is a solution of (1.1) satisfying (3.1), then we have*

$$ru_r \to -\beta \quad \text{and} \quad u_\theta \to 0 \quad \text{as} \quad r \to \infty.$$

Proof. Based on previous two lemmas and the lower estimate $\beta > \max\{2(N+1), 2/a\}$, the proof of Lemma 3.3 follows exactly in the same way as the proof of Lemma 1.3 in [11], by replacing the term $R(y)e^{u(y)}$ with $(|y|^{2N}e^{u(y)} + \lambda e^{au(y)})$. Here we omit the details of the proof. □

We are now ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We are only left to show the energy identity (1.9) in view of Lemma 3.2. Multiplying both hand sides of equation (1.1) by $x \cdot \nabla u$ and integrating by parts over the ball $B_K(0)$, one gets

$$\begin{aligned}
&\int_{\partial B_K(0)} r\Big(\frac{1}{2}|\nabla u|^2 - u_r^2\Big)\, ds \\
&= -\int_{B_K(0)} (x \cdot \nabla |x|^{2N}) e^u \, dx - 2\int_{B_K(0)} |x|^{2N} e^u \, dx \\
&\quad + \int_{\partial B_K(0)} r|x|^{2N} e^u \, ds - 2\int_{B_K(0)} \frac{\lambda}{a} e^{au} \, dx + \int_{\partial B_K(0)} r\frac{\lambda}{a} e^{au} \, ds \\
&= -\int_{B_K(0)} \Big[2N|x|^{2N}e^u + \Big(\frac{2}{a} - 2\Big)\lambda e^{au}\Big] \, dx \\
&\quad - 2\int_{B_K(0)} \Big[|x|^{2N}e^u + \lambda e^{au}\Big] \, dx + \int_{\partial B_K(0)} r\Big(|x|^{2N}e^u + \frac{\lambda}{a}e^{au}\Big) \, ds,
\end{aligned}$$

where $|\nabla u|^2 = u_r^2 + \frac{1}{r^2}u_\theta$. Note from Lemma 3.3 that

$$\int_{\partial B_K(0)} r\Big(\frac{1}{2}|\nabla u|^2 - u_r^2\Big)\, ds \to -\pi\beta^2 \quad \text{as} \quad K \to \infty.$$

On the other hand, it yields from Lemma 3.2 that

$$\int_{\partial B_K(0)} r\Big(|x|^{2N}e^u + \frac{\lambda}{a}e^{au}\Big) \, ds \to 0 \quad \text{as} \quad K \to \infty.$$

Therefore, we conclude that

$$\pi\beta^2 = \int \Big[2N|x|^{2N}e^u + \Big(\frac{2}{a} - 2\Big)\lambda e^{au}\Big] \, dx + 4\pi\beta,$$

which implies (1.9), and hence Theorem 1.2 follows. □

4. Refined Asymptotic Behavior

This section is focused on the proofs of Theorem 1.3 and Proposition 1.4, which handle the refined asymptotic behavior of condensate solutions of (1.1) as $|x| \to \infty$. The basic tool is to reduce (1.1) into semilinear evolution elliptic problems.

In this section, let $u = u(x)$ be a fixed condensate solution of (1.1) satisfying (1.7). As mentioned in the introduction, by applying the estimate (1.8), standard elliptic theory implies that any solution of (1.1) satisfying (1.7) must be a classical solution. Thus, there exists some function $\psi(\cdot) \in C^0(S^1)$ such that u satisfies

$$\begin{cases} -\Delta u = \lambda e^{au} + |x|^{2N} e^u & \text{in} \quad \mathbb{R}^2 \setminus B_1(0), \\ u = \psi(\cdot) & \text{on} \quad S^1 = \partial B_1(0), \end{cases} \tag{4.1}$$

where $B_1(0)$ is the unit ball in $\mathbb{R}^2$ with the center at zero. Using the idea from [13] and the references therein, we set

$$v(t,\theta) := u(r,\theta) + \beta \ln r = u(x) + \beta \ln |x|, \quad t = \ln r \text{ and } r = |x|, \tag{4.2}$$

where β is as in (1.7) and $(r,\theta) \in (1,\infty) \times S^1$ are polar coordinates in $\mathbb{R}^2 \setminus B_1(0)$, to obtain an evolution elliptic equation of the form

$$\begin{cases} -v_{tt} - v_{\theta\theta} = \lambda e^{(2-a\beta)t} e^{av} + e^{(2N+2-\beta)t} e^v, & (t,\theta) \in (0,\infty) \times S^1, \\ v(0,\cdot) = \psi(\cdot), \end{cases} \tag{4.3}$$

where $\psi(\cdot)$ is the same as that in (4.1), and Theorem 1.2 implies that $2 - a\beta < 0$ and $2(N+1) - \beta < 0$. Therefore, discussing the asymptotic behavior of u as $|x| \to \infty$ is reduced into studying the large-time behavior of bounded solution v of (4.3) as $t \to +\infty$.

Remark 4.1. Similar to Sections 1 and 2 in [13] (and many related papers), the use of such terminology as “orbit" or “ω-limit set" below is simply for the sake of convenience, and it does not mean that the set is an orbit of some dynamical system in the correct sense. In fact, as pointed out by the referee, we do not know whether some local semiflow can be defined in our problem. It is therefore simply for the sake of convenience that we borrow some terminology from the theory of dynamical systems in this whole section.

Lemma 4.1. *Under the transformation (4.2), the following results hold:*

(1) *There exists some $\delta \in (0,1)$ such that $v(t,\cdot)$, $v_t(t,\cdot)$, $v_\theta(t,\cdot)$, $v_{tt}(t,\cdot)$, $v_{t\theta}(t,\cdot)$, $v_{\theta\theta}(t,\cdot)$, $v_{ttt}(t,\cdot)$, $v_{t\theta\theta}(t,\cdot)$, $v_{tt\theta}(t,\cdot)$ and $v_{\theta\theta\theta}(t,\cdot)$ all remain bounded in $C^\delta(S^1)$ for any $t \in [0,+\infty)$, where $C^\delta(S^1)$ denotes the usual Hölder continuous space on S^1.*
(2) *Both $v_t(t,\cdot)$ and $v_{tt}(t,\cdot)$ tend to 0 in $C^0(S^1)-$topology as $t \to +\infty$.*
(3) *The “orbit" $\mathcal{L} := \{v(t,\cdot;\psi) : t \geq 0\}$ of v is relatively compact in $C^3(S^1)$.*

Proof. Since Theorem 1.2 gives that v is a uniformly bounded solution of (4.3) with bounded coefficients, we first note that (1) is an immediate consequence of L^p and Schauder estimates for (4.3).

To prove (2), multiplying (4.3) by v_t and integrating it by parts in θ and t, and then using the boundedness of v, v_θ and v_t, one can obtain that

$$\int_0^\infty \int_{S^1} v_t^2 d\theta dt < +\infty. \tag{4.4}$$

Define

$$k(t) = \int_{S^1} v_t^2 d\theta.$$

The boundedness of v_t and v_{tt} implies that $k'(t)$ is bounded on $[0,\infty)$, and furthermore, (4.4) implies that

$$k(t) \geq 0, \quad \int_0^\infty k(t)dt < +\infty.$$

and hence $k(t) \to 0$ as $t \to +\infty$. It then follows from the boundedness of $v_{t\theta}$ that

$$v_t \to 0 \quad \text{as} \quad t \to +\infty \quad (\text{uniformly in } \theta \in S^1). \tag{4.5}$$

Therefore, the convergence $v_{tt} \to 0$, as $t \to +\infty$, follows from (4.5) and the boundedness of v_{tt}, v_{ttt} and $v_{tt\theta}$.

Finally, (3) follows from (1) and the Ascoli-Arzelá Theorem. □

In view of Lemma 4.1, we next define "$\omega-$limit set" $\Gamma(\mathcal{L})$ of $\mathcal{L} := \{v(t, \cdot; \psi) : t \geq 0\}$ as

$$\Gamma(\mathcal{L}) := \bigcap_{t\geq 0} \text{closure}\{v(\iota, \cdot);\, \iota \geq t\}, \tag{4.6}$$

where the closure is with respect to the topology of $C^2(S^1)$. A standard dynamical systems argument shows that $\Gamma(\mathcal{L})$ is a nonempty compact connected set in $C^2(S^1)$. Moveover, since $2 - a\beta < 0$ and $2(N + 1) - \beta < 0$, Lemmas 3.3 and 4.1 imply that $\Gamma(\mathcal{L}) \subset \mathfrak{S}$, where $\mathfrak{S}$ is the set of stationary solutions of (4.3), $i.e.$,

$$\mathfrak{S} := \{w(\theta) \in C^2(S^1) : w'' = w' = 0\} \subset \mathbb{R}, \tag{4.7}$$

and $w_\theta = 0$ is followed from Lemma 3.3. Furthermore, the convergence of $w(t, \cdot)$ as $t \to \infty$ can be established by integrating (4.3) over S^1, together with the exponent decay property of the right-hand side terms of (4.3). Therefore, the above analysis leads to the following asymptotic behavior.

Theorem 4.2. *Let u be a solution of (1.1) satisfying (1.7), then there exists a constant $K = K\big(\lambda, a, N, \beta, u(0)\big) \in \mathbb{R}$ such that*

$$u(x) + \beta \ln |x| \to K \quad \textit{as} \quad |x| \to \infty, \tag{4.8}$$

where $\beta > \max\big\{\frac{2}{a}, 2(N + 1)\big\}$ is as in (1.7).

4.1. **Further analysis of refined asymptotic behavior.** In this subsection, we introduce a new transformation

$$V(t, \theta) = u(r, \theta) + \beta \ln r - K \quad \text{in} \quad (0, \infty) \times S^1, \tag{4.9}$$

so that $\lim_{t\to+\infty} V(t, \cdot) = 0$, where the constant $\beta > \max\big\{\frac{2}{a}, 2(N+1)\big\}$ is as in (1.7), and the constant $K = K\big(\lambda, a, N, \beta, u(0)\big)$ is as in Theorem 4.2. Thus, $V(t, \theta)$ is a uniformly bounded solution of the following evolution elliptic equation

$$-V_{tt} - V_{\theta\theta} = \lambda e^{aK+(2-a\beta)t} e^{aV} + e^{K+(2N+2-\beta)t} e^{V}, \quad (t, \theta) \in (0, \infty) \times S^1. \tag{4.10}$$

See [13] for other evolution elliptic equations applied to elliptic PDEs. To complete the proofs of Theorem 1.3 and Proposition 1.4, we shall carry on a delicate study of the asymptotic behavior of Fourier coefficients of $V(t, \theta)$ satisfying (4.10). We start with the following estimate.

Lemma 4.3. *Let $V(t, \theta)$ satisfy (4.9), then*

$$\sup_{t\geq 0} e^{\varepsilon t} \|V(t, \cdot)\|_{C^0(S^1)} < +\infty \tag{4.11}$$

holds for some constant $0 < \varepsilon \leq \tau := \min\big\{a\beta - 2, \beta - 2(N + 1)\big\}$.

Proof. On the contrary, suppose that (4.11) is false. Set $\rho(t) = \|V(t, \cdot)\|_{C^0(S^1)}$, then $\rho(t) \in C^0\big([0, \infty)\big)$, and

$$\lim_{t\to\infty} \rho(t) = 0, \quad \lim_{t\to\infty} \sup e^{\varepsilon t} \rho(t) = +\infty \tag{4.12}$$

for any constant $0 < \varepsilon \leq \tau := \min\big\{a\beta - 2, \beta - 2(N + 1)\big\}$. By Lemma A.1 in [13], there exists a function $\eta(t) \in C^\infty\big([0, \infty)\big)$ such that

$$\eta(t) > 0, \quad \eta'(t) < 0, \quad \lim_{t\to\infty} \eta(t) = 0, \tag{4.13}$$

$$0 < \lim_{t\to\infty} \sup \frac{\rho(t)}{\eta(t)} < +\infty, \tag{4.14}$$

$$\lim_{t\to\infty} e^{\varepsilon t}\eta(t) = +\infty, \tag{4.15}$$

$$\big(\eta'/\eta\big)',\ \big(\eta''/\eta\big)' \in L^1\big((0,\infty)\big), \tag{4.16}$$

$$\lim_{t\to\infty} \frac{\eta'(t)}{\eta(t)} = \lim_{t\to\infty} \frac{\eta''(t)}{\eta(t)} = 0. \tag{4.17}$$

Define $w(t,\theta) = \frac{V(t,\theta)}{\eta(t)}$, then w is bounded in $[0,\infty)\times S^1$, and (4.10) implies that w satisfies

$$w_{tt} + 2\frac{\eta'}{\eta}w_t + \frac{\eta''}{\eta}w + w_{\theta\theta} + \frac{\lambda}{\eta}e^{aK+(2-a\beta)t}(e^{aw})^{\eta} + \frac{1}{\eta}e^{K+(2N+2-\beta)t}(e^{w})^{\eta} = 0, \tag{4.18}$$

where $(t,\theta)\in(0,\infty)\times S^1$, and the constant $K = K\big(\lambda, a, N, \beta, u(0)\big)$ is as in Theorem 4.2. Note from (4.15) that

$$\lim_{t\to\infty} \frac{\lambda}{\eta(t)}e^{(2-a\beta)t} = \lim_{t\to\infty} \frac{1}{\eta(t)}e^{(2N+2-\beta)t} = 0.$$

Thus, similar to (1) of Lemma 4.1, by applying L^p and Schauder estimates for (4.18) about a linear elliptic equation with bounded coefficients, one can deduce that for some $\delta\in(0,1)$, $w(t,\cdot)$, $w_t(t,\cdot)$, $w_\theta(t,\cdot)$, $w_{tt}(t,\cdot)$, $w_{t\theta}(t,\cdot)$, $w_{\theta\theta}(t,\cdot)$, $w_{ttt}(t,\cdot)$, $w_{t\theta\theta}(t,\cdot)$, $w_{tt\theta}(t,\cdot)$ and $w_{\theta\theta\theta}(t,\cdot)$ all remain bounded in $C^\delta(S^1)$ for any $t\in[0,+\infty)$. Applying (4.16) and (4.17), as in (2) of Lemma 4.1, one can further prove that $w_t(t,\cdot)$ and $w_{tt}(t,\cdot)$ tend to 0 in $C^0(S^1)-$topology as $t\to+\infty$. So if we define "$\omega-$limit set" $\Gamma(\mathcal{L}')$ of the "orbit" $\mathcal{L}' := \{w(t,\cdot) : t\geq 0\}$ for (4.18), as

$$\Gamma(\mathcal{L}') := \bigcap_{t\geq 0} \text{closure}\{w(\iota,\cdot);\ \iota\geq t\},$$

where the closure is with respect to the topology of $C^2(S^1)$, then a standard argument of dynamical systems shows that $\Gamma(\mathcal{L}')$ is a nonempty compact connected set in $C^2(S^1)$. Moreover, $\Gamma(\mathcal{L}')\subset\mathfrak{S}'$, where $\mathfrak{S}'$ is the set of stationary solutions of (4.18), *i.e.*,

$$\mathfrak{S}' := \{w(\theta)\in C^2(S^1) : w'' = 0\}\subset\mathbb{R},$$

since w is bounded in S^1.

Because (4.14) implies $0\notin\Gamma(\mathcal{L}')$, we now assume that there exists a constant $\kappa\neq 0$ such that $\kappa\in\Gamma(\mathcal{L}')$. This implies that there exists a sequence $\{t_i\}_{i=1}^\infty$ satisfying $\|w(t_i,\theta)\|_{C^0(S^1)}\to\kappa\neq 0$ as $i\to\infty$. Define

$$f(t) = \kappa\int_{S^1} V(t,\theta)d\theta. \tag{4.19}$$

Since $\eta(t) > 0$ and $w(\theta)\equiv\kappa\in\Gamma(\mathcal{L}')$ for any $\theta\in S^1$, we first note that there exists a sufficiently large $i_0\in\mathbb{N}$ such that $f(t_i) > 0$ for any $i\geq i_0$. On the other hand, multiplying (4.10) by κ and integrating it over S^1, we obtain that

$$-f''(t) = \kappa\lambda e^{aK}\int_{S^1} e^{(2-a\beta)t}e^{aV}d\theta + \kappa e^K\int_{S^1} e^{(2N+2-\beta)t}e^V d\theta. \tag{4.20}$$

Since $f(t_i) > 0$ for any $i\geq i_0$, and $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = 0$, it follows from (4.20) that $\kappa < 0$. Thus, we further derive from (4.20) that

$$-f''(t)\geq\kappa C_1 e^{(2-a\beta)t} + \kappa C_2 e^{(2N+2-\beta)t}\geq C\kappa e^{-\varepsilon t}$$

for all $t = t_i$ with $i \geq i_0$, since $\varepsilon \leq \tau$. It hence follows from the above that

$$0 \leq f(t_i) \leq \frac{C|\kappa|}{\varepsilon^2} e^{-\varepsilon t_i} \quad \text{for any } i \geq i_0. \tag{4.21}$$

Therefore, this estimate and (4.15) yield that

$$\lim_{i\to\infty} \sup \frac{f(t_i)}{\eta(t_i)} \leq \frac{\lim_{i\to\infty} \sup Ce^{\varepsilon t_i} f(t_i)}{\lim_{i\to\infty} \inf Ce^{\varepsilon t_i} \eta(t_i)} = 0,$$

which then implies that $0 \in \Gamma(\mathcal{L}')$, a contradiction. □

The following Fourier analysis gives essentially a better idea of estimating the power ε of (4.11).

Lemma 4.4. *Let $V(t,\theta)$ satisfy (4.9), then there exists some constant $M > 0$ such that*

$$\|V(t,\cdot)\|_{C^0(S^1)} \leq Me^{-\Gamma t}, \tag{4.22}$$

where $\Gamma > 0$ satisfies
(1). $\Gamma = \min\{1, \tau\}$, if $\tau \notin \mathbb{N}$;
(2). $\Gamma = \min\{1, \tau - \epsilon\}$ with an arbitrary $\epsilon > 0$, if $\tau \in \mathbb{N}$.

Proof. Consider Fourier series of V, e^{aV} and e^V:

$$\begin{aligned} V(t,\theta) &= (2\pi)^{-\frac12} \textstyle\sum_{k\in\mathbb{Z}} a_k(t) e^{ik\theta}, \\ e^{aV(t,\theta)} &= (2\pi)^{-\frac12} \textstyle\sum_{k\in\mathbb{Z}} A_k(t) e^{ik\theta}, \quad e^{V(t,\theta)} = (2\pi)^{-\frac12} \textstyle\sum_{k\in\mathbb{Z}} B_k(t) e^{ik\theta}. \end{aligned} \tag{4.23}$$

By Lemma 4.3, $w(t,\theta) = e^{\varepsilon t} V(t,\theta)$ is uniformly bounded in $[0,\infty) \times S^1$, where $0 < \varepsilon \leq \tau := \min\big\{a\beta - 2, \beta - 2(N+1)\big\}$ is the same as that of Lemma 4.3. (4.10) then implies that w satisfies a new evolution elliptic equation

$$\begin{aligned} -w_{tt} + 2\varepsilon w_t - \varepsilon^2 w - w_{\theta\theta} &= \lambda e^{aK + (2 - a\beta + \varepsilon)t} \big(e^{aw}\big)^{e^{-\varepsilon t}} \\ &\quad + e^{K + (2N+2-\beta+\varepsilon)t} \big(e^{w}\big)^{e^{-\varepsilon t}} \qquad \text{in } (0,\infty) \times S^1, \end{aligned} \tag{4.24}$$

where $2 - a\beta + \varepsilon \leq 0$, $2N + 2 - \beta + \varepsilon \leq 0$, and the constant $K = K\big(\lambda, a, N, \beta, u(0)\big)$ is as in Theorem 4.2. As before, a priori estimates show that w and its derivatives, up to the third order, remain bounded on $[0,\infty) \times S^1$, $i.e$,

$$\|e^{\varepsilon t} V(t,\theta)\|_{C^3([0,\infty)\times S^1)} < +\infty. \tag{4.25}$$

It then follows that there exists $M_1 > 0$ such that

$$\begin{aligned} &\textstyle\sum_{k\in\mathbb{Z}} (k^2+1)|a_k(t)|^2 \leq M_1 e^{-2\varepsilon t}, \\ &\textstyle\sum_{k\in\mathbb{Z}} (k^2+1)|A_k(t)|^2 \leq M_1, \quad \textstyle\sum_{k\in\mathbb{Z}} (k^2+1)|B_k(t)|^2 \leq M_1 \end{aligned} \tag{4.26}$$

for all $t \geq 0$. It also follows from (4.10) that $a_k(t)$ is a bounded solution of the ODE

$$a_k''(t) - k^2 a_k(t) = g_k(t) := -\lambda e^{aK + (2-a\beta)t} A_k(t) - e^{K + (2N+2-\beta)t} B_k(t), \tag{4.27}$$

where $g_k(t)$ satisfies

$$|g_k(t)| \leq \frac{C_1}{\sqrt{k^2+1}} e^{-\tau t}$$

for some positive constant C_1 by applying (4.26). The integration of (4.27) yields that

$$\begin{aligned} a_0(t) &= -\int_t^{\infty}(t-s)g_0(s)ds,\\ a_k(t) &= a_k(0)e^{-|k|t}+\frac{1}{2|k|}e^{-|k|t}\int_0^t\big(e^{-|k|s}-e^{|k|s}\big)g_k(s)ds \\ &\quad +\frac{1}{2|k|}\big(e^{-|k|t}-e^{|k|t}\big)\int_t^{\infty}e^{-|k|s}g_k(s)ds,\quad |k|\geq 1. \end{aligned} \tag{4.28}$$

This implies that there exists a sequence of nonnegative constants $\{P_k\}_{|k|=1}^{\infty}$ such that

$$\begin{aligned} |a_0(t)| &\leq -C_1\int_t^{\infty}(t-s)e^{-\tau s}ds=\frac{C_1}{\tau^2}e^{-\tau t}\leq M_2e^{-\tau t},\\ |a_k(t)| &\leq P_ke^{-|k|t}+\frac{M_2}{|k|\sqrt{k^2+1}}e^{-\tau t},\quad \text{if } |k|\geq 1 \text{ and } \tau\notin\mathbb{N},\\ |a_{\pm\tau}(t)| &\leq P_{\pm\tau}e^{-\tau t}+M_2\big(1+t\big)e^{-\tau t},\quad \text{if}\quad \tau\in\mathbb{N}, \end{aligned} \tag{4.29}$$

where $M_2>0$ is a constant.

It now follows from (4.29) that if $\tau\notin\mathbb{N}$, then there exists $M_3>0$ such that

$$\|V(t,\cdot)\|^2_{H^1(S^1)}=\sum_{k\in\mathbb{Z}}(k^2+1)|a_k(t)|^2\leq M_3e^{-2\Gamma t},\quad \text{where}\quad \Gamma=\min\{1,\tau\},$$

which further implies that

$$\|V(t,\cdot)\|_{C^0(S^1)}\leq Me^{-\Gamma t}$$

holds for some constant $M>0$. This gives (1) of Lemma 4.4. Similarly, one can follow (4.29) to obtain (2) of Lemma 4.4 for the case $\tau\in\mathbb{N}$. □

Remark 4.2. Let n_0 be the largest integer smaller than $\tau:=\min\big\{a\beta-2,\beta-2(N+1)\big\}$. Then for any $\mathrm{n}\in\mathbb{N}$ satisfying $0\leq\mathrm{n}\leq\mathrm{n}_0$, if

$$\lim_{t\to\infty}\|V(t,\cdot)\|_{C^0(S^1)}e^{\mathrm{n}t}=0, \tag{4.30}$$

then there exists some constant $M>0$ such that

$$\|V(t,\cdot)\|_{C^0(S^1)}\leq Me^{-pt}, \tag{4.31}$$

where $p>0$ satisfies

(1) $p=\min\{\mathrm{n}+1,\tau\}$, if $\tau\notin\mathbb{N}$;
(2) $p=\min\{\mathrm{n}+1,\tau-\epsilon\}$ with an arbitrary $\epsilon>0$, if $\tau\in\mathbb{N}$.

Indeed, if (4.30) holds for some integer n satisfying $0\leq\mathrm{n}\leq\mathrm{n}_0$, then Lemma 4.3 gives that $\sup_{t\geq 0}e^{\varepsilon t}\|V(t,\cdot)\|_{C^0(S^1)}<+\infty$ holds for some ε satisfying $\mathrm{n}<\varepsilon\leq\tau$. Based on this fact, a similar proof of Lemma 4.4 then yields the conclusion (4.31).

Proposition 4.5. *Let $V(t,\theta)$ satisfy (4.9), and suppose that there exists a constant $0<\gamma\leq\tau:=\min\big\{a\beta-2,\beta-2(N+1)\big\}$ such that*

$$\|V(t,\cdot)\|_{C^0(S^1)}\leq Ce^{-\gamma t} \tag{4.32}$$

holds for some constant $C>0$. Then we have the following conclusions:

(A). If $\gamma<\tau$ and γ is a natural number, then there exist $A_\gamma\in\mathbb{R}$ and $\theta_\gamma\in S^1$ such that

$$V(t,\theta)\ \sim\ e^{-\gamma t}A_\gamma\sin(\gamma\theta+\theta_\gamma)\quad as\quad t\to\infty.$$

(B). If $\gamma=\tau$, then there exist $A\in\mathbb{R}$ and $\theta_0\in S^1$ such that

$$V(t,\theta)\ \sim\ e^{-\tau t}\big[A\sin(\tau\theta+\theta_0)-M\big]\quad as\quad t\to\infty,$$

where $A=0$ for the case $\gamma=\tau\notin\mathbb{N}$, τ and M satisfy

(i). for $\beta - 2N > a\beta$, $\tau = a\beta - 2$ and $M = \frac{\lambda e^{aK}}{(a\beta-2)^2}$;
(ii). for $\beta - 2N = a\beta$, $\tau = a\beta - 2$ and $M = \frac{\lambda e^{aK}+e^K}{(a\beta-2)^2}$;
(iii). for $\beta - 2N < a\beta$, $\tau = \beta - 2N - 2$ and $M = \frac{e^K}{(\beta-2N-2)^2}$.

Proof. Define

$$w(t,\theta) = e^{\gamma t} V(t,\theta), \tag{4.33}$$

where $\gamma > 0$ is the same as that of (4.32). It then follows from (4.32) that w is uniformly bounded in $[0,\infty) \times S^1$ and satisfies

$$\begin{aligned} -w_{tt} + 2\gamma w_t - \gamma^2 w - w_{\theta\theta} &= \lambda e^{aK+(2-a\beta+\gamma)t}\big(e^{aw}\big)^{e^{-\gamma t}} \\ &\quad + e^{K+(2N+2-\beta+\gamma)t}\big(e^{w}\big)^{e^{-\gamma t}} \qquad \text{in } (0,\infty)\times S^1. \end{aligned} \tag{4.34}$$

Again, standard L^p and Schauder estimates of (4.34) imply that w and its derivatives, up to the third order, remain bounded on $[0,\infty)\times S^1$. By integration by parts, one can further prove that $w_t(t,\cdot)$ and $w_{tt}(t,\cdot)$ tend to 0 in $C^0(S^1)-$topology as $t \to +\infty$. So if we define "$\omega-$limit set" $\Gamma(\mathcal{L}'')$ of the "orbit" $\mathcal{L}'' := \{w(t,\cdot) : t \geq 0\}$ for (4.34), as

$$\Gamma(\mathcal{L}'') := \bigcap_{t\geq 0} \text{closure}\{w(\iota,\cdot);\ \iota \geq t\},$$

where the closure is with respect to the topology of $C^2(S^1)$, then we have again that $\Gamma(\mathcal{L}'')$ is a nonempty compact connected set in $C^2(S^1)$. Moreover, $\Gamma(\mathcal{L}'') \subset \mathfrak{S}''$, where $\mathfrak{S}''$ is the set of stationary solutions of (4.34).

(A). If $0 < \gamma < \tau$ and γ is a natural number, then $\mathfrak{S}''$ is a nonempty compact connected subset of

$$\Big\{\psi(\theta) \in C^2(S^1) : \frac{d^2\psi}{d\theta^2} + \gamma^2\psi = 0\Big\}, \tag{4.35}$$

and the convergence of $w(t,\cdot)$ as $t \to \infty$ follows from [17]. We next prove that $w(t,\cdot)$ has the exponential convergence as $t \to \infty$: Consider the bounded Fourier coefficients $a_k(t)$ of $w(t,\theta)$, which is defined by

$$a_k(t) := (2\pi)^{-1/2} \int_{S^1} w(t,\theta) e^{-ik\theta} d\theta, \quad k \in \mathbb{Z}.$$

One can deduce from (4.34) that the uniformly bounded function $a_k(t)$ satisfies the ODE

$$a_k''(t) - 2\gamma a_k'(t) + (\gamma^2 - k^2) a_k(t) = e^{-\mu t} F(t), \tag{4.36}$$

where $\mu = \tau - \gamma > 0$, and $F(t)$ is uniformly bounded in $[0,\infty)$. Because of the uniform boundedness of $a_k(t)$, the ODE analysis shows that

$$\begin{aligned} a_0(t) &= e^{\gamma t} \int_\infty^t (t-s) e^{-(\mu+\gamma)s} F(s) ds, \\ a_k(t) &= \frac{1}{2|k|} e^{(\gamma-|k|)t} \int_\infty^t \big[e^{2|k|t} - e^{2|k|s}\big] e^{-(\mu+\gamma+|k|)s} F(s) ds, \quad 1 \leq |k| \leq \gamma, \\ a_k(t) &= a_k(0) e^{(\gamma-|k|)t} + \frac{1}{2|k|} e^{(\gamma-|k|)t} \int_0^t \big[e^{-(\gamma+|k|)s} - e^{-(\gamma-|k|)s}\big] e^{-\mu s} F(s) ds \\ &\quad - \frac{1}{2|k|} \big[e^{(\gamma-|k|)t} - e^{(\gamma+|k|)t}\big] \int_\infty^t e^{-(\gamma+|k|)s} e^{-\mu s} F(s) ds, \quad |k| > \gamma, \end{aligned}$$

where $\gamma - |k| + \mu = \tau - |k|$. Delicate estimates yield from the above that there exist two positive constants C_1 and C_2 such that

$$|a_0(t)| \le C_2 e^{-\mu t}, \quad 0 \le |k| \le \gamma,$$

$$|a_k(t)| \le C_1(1+t)e^{(\gamma-|k|)t} + \frac{C_2}{|k|\sqrt{k^2+1}}e^{-\mu t}, \quad |k| > \gamma.$$

Thus, the exponential decay property of $a_k(t)$ implies the exponential convergence of $w(t,\cdot)$ as $t \to \infty$.

Therefore, it now follows from (4.35) that there exist $A_\gamma \in \mathbb{R}$ and $\theta_\gamma \in S^1$ such that

$$w(\theta) = A_\gamma \sin(\gamma\theta + \theta_\gamma) \in \Gamma(\mathcal{L}''), \quad \text{where} \quad \theta \in S^1, \tag{4.37}$$

and hence (A) of Proposition 4.5 follows from (4.33) and (4.37).

(B). If $\gamma = \tau$, it then follows from (4.34) that $\mathfrak{S}''$ is a nonempty compact connected subset of

$$\Big\{\psi(\theta) \in C^2(S^1) : -\frac{d^2\psi}{d\theta^2} - \gamma^2\psi = C\Big\}, \tag{4.38}$$

where the constant C is explicitly given below. Furthermore, if the convergence of $w(t,\cdot)$ holds, it then follows from (4.38) that there exist $A \in \mathbb{R}$ and $\theta_0 \in S^1$ such that

(1) If $\beta - 2N > a\beta$, then $\gamma = a\beta - 2$, $C = \lambda e^{aK}$ in (4.38), and

$$w(\theta) = A\sin[(a\beta-2)\theta + \theta_0] - \frac{\lambda e^{aK}}{(a\beta-2)^2} \in \Gamma(\mathcal{L}''), \quad \text{where} \quad \theta \in S^1. \tag{4.39}$$

(2) If $\beta - 2N = a\beta$, then $\gamma = a\beta - 2$, $C = \lambda e^{aK} + e^K$ in (4.38), and

$$w(\theta) = A\sin[(a\beta-2)\theta + \theta_0] - \frac{\lambda e^{aK} + e^K}{(a\beta-2)^2} \in \Gamma(\mathcal{L}''), \quad \text{where} \quad \theta \in S^1. \tag{4.40}$$

(3) If $\beta - 2N < a\beta$, then $\gamma = \beta - 2(N+1)$, $C = e^K$ in (4.38), and

$$w(\theta) = A\sin[(\beta-2N-2))\theta+\theta_0] - \frac{e^K}{(\beta-2N-2)^2} \in \Gamma(\mathcal{L}''), \quad \text{where} \quad \theta \in S^1. \tag{4.41}$$

Here $A = 0$ for the case $\gamma = \tau \notin \mathbb{N}$, and the constant $K = K\big(\lambda, a, N, \beta, u(0)\big)$ is as in Theorem 4.2.

Next, we prove the convergence of $w(t,\cdot)$ as $t \to \infty$. We only focus on the above case (1), where $\beta - 2N > a\beta$, $\gamma = \tau = a\beta - 2$ and $C = \lambda e^{aK}$ in (4.38), since the other two cases can be addressed similarly. Set $W(t,\theta) := w(t,\theta) + \frac{\lambda e^{aK}}{\tau^2}$, then W is a uniformly bounded solution of

$$\begin{aligned}
-W_{tt} + 2\tau W_t - \tau^2 W - W_{\theta\theta} =\ & \lambda e^{aK}\big[\big(e^{a(W-\frac{\lambda e^{aK}}{\tau^2})}\big)^{e^{-\tau t}} - 1\big] \\
& + e^{K+(2N+2-\beta+\tau)t}\big(e^{W-\frac{\lambda e^{aK}}{\tau^2}}\big)^{e^{-\tau t}} \\
:=\ & e^{-pt}f(t,\theta) \quad \text{in } (0,\infty)\times S^1,
\end{aligned} \tag{4.42}$$

where $p = \min\{\tau, \beta - 2N - 2 - \tau\} > 0$ in this case, and $f(t,\theta)$ is uniformly bounded in $[0,\infty)\times S^1$. Because the inhomogeneous term of (4.42) decays exponentially, applying [17] or a similar proof of (A) above yields also that all Fourier coefficients of $W(t,\theta)$ decay exponentially, which further implies the (exponential) convergence of $W(t,\cdot)$ and hence $w(t,\cdot)$ as $t \to \infty$.

Therefore, (B) of Proposition 4.5 follows from (4.33), and (4.39)–(4.41). □

We are now ready to complete the proof of Theorem 1.3 by applying Lemma 4.4 and Proposition 4.5.

Proof of Theorem 1.3. (I). We first consider the case $\tau \notin \mathbb{N}$ to prove (I) of Theorem 1.3.

If $0 < \tau := \min\big\{a\beta - 2, \beta - 2(N+1)\big\} < 1$, then Lemma 4.4 holds with $\Gamma = \tau$, and hence applying (B) of Proposition 4.5 implies that (2) of Theorem 1.3 occurs necessarily.

We next consider the case $1 < \tau < 2$. In this case, Lemma 4.4 holds with $\Gamma = 1$, and hence (A) of Proposition 4.5 occurs at $\gamma = 1$ with some $A_1 \in \mathbb{R}$. Note that if $A_1 \neq 0$ occurs here, then (1) of Theorem 1.3 occurs with $k_0 = 1$.

Otherwise, if $A_1 = 0$ occurs here, then $\|V(t,\theta)\|_{C^0(S^1)}e^t \to 0$ as $t \to \infty$. Thus, Remark 4.2 yields that there exists some constant $M > 0$ such that

$$\|V(t,\cdot)\|_{C^0(S^1)} \leq Me^{-\tau t}, \tag{4.43}$$

since $1 < \tau < 2$. Hence applying (B) of Proposition 4.5 implies that (2) of Theorem 1.3 occurs. This proves the case $1 < \tau < 2$.

For the general case $2 \leq \mathrm{n}_0 < \tau < \mathrm{n}_0 + 1$, where n_0 is an integer, one can note that if (A) of Proposition 4.5 occurs at some natural number $\gamma < \tau$ with $A_\gamma = 0$, then $\|V(t,\theta)\|_{C^0(S^1)}e^{\gamma t} \to 0$ as $t \to \infty$. Therefore, one can obtain from Remark 4.2 that $\|V(t,\cdot)\|_{C^0(S^1)} \leq Me^{-pt}$, where $p = \min\{\gamma+1, \tau\}$. Based on this fact, the basic idea of the proof is similar to that of the above case where $1 < \tau < 2$, and we omit the details.

(II). We now consider the case $\tau \in \mathbb{N}$ to prove (II) of Theorem 1.3.

If $\tau = 2$, then Lemma 4.4 holds with $\Gamma = 1$, and hence (A) of Proposition 4.5 occurs at $\gamma = 1$ with some $A_1 \in \mathbb{R}$.

If $\tau = 3$, then Lemma 4.4 also holds with $\Gamma = 1$, and hence (A) of Proposition 4.5 occurs at $\gamma = 1$ with some $A_1 \in \mathbb{R}$. If $A_1 \neq 0$, then the proof is completed. Otherwise, if $A_1 \neq 0$ occurs here, then $\|V(t,\theta)\|_{C^0(S^1)}e^t \to 0$ as $t \to \infty$. So Remark 4.2 yields that there exists some constant $M > 0$ such that

$$\|V(t,\cdot)\|_{C^0(S^1)} \leq Me^{-2t}. \tag{4.44}$$

Hence (A) of Proposition 4.5 occurs at $\gamma = 2$ with some $A_2 \in \mathbb{R}$, and we are done.

For the general case $\tau \geq 4$, one can note as before that if (A) of Proposition 4.5 occurs at some natural number $\gamma < \tau - 1$ with $A_\gamma = 0$, then $\|V(t,\theta)\|_{C^0(S^1)}e^{\gamma t} \to 0$ as $t \to \infty$. Therefore, one can obtain from Remark 4.2 that $\|V(t,\cdot)\|_{C^0(S^1)} \leq Me^{-pt}$, where $p = \gamma + 1 < \tau$. Based on this fact, the idea of the proof is similar to that of the above case where $\tau = 3$, and we omit the details. □

For any radial condensate solution of (1.1) satisfying (1.7), we next prove Proposition 1.4 to obtain the more precise asymptotic behavior.

Proof of Proposition 1.4. Suppose that $u = u(r)$ is a radial condensate solution of (1.1) satisfying (1.7), Theorem 4.2 then implies that there exists a constant $K \in \mathbb{R}$ such that

$$u(r) + \beta \ln r \to K \quad \text{as} \quad r \to \infty,$$

where $\beta > \max\big\{\frac{2}{a}, 2(N+1)\big\}$ is as in (1.7). Further, it follows from Lemma 4.4 and Proposition 4.5 that there exists a constant $M > 0$ such that

$$|V(t)| := |u(r) + \beta \ln r - K| \leq Me^{-\tau t}, \tag{4.45}$$

where $\tau := \min\big\{a\beta - 2, \beta - 2(N+1)\big\}$. Set

$$w(t) = e^{\tau t}V(t). \tag{4.46}$$

It then follows from (4.10) and (4.45) that w is uniformly bounded in $[0,\infty)$ and satisfies

$$\begin{aligned} -w'' + 2\tau w' - \tau^2 w = \lambda e^{aK+(2-a\beta+\tau)t}\big(e^{aw}\big)^{e^{-\tau t}} \\ + e^{K+(2N+2-\beta+\tau)t}\big(e^{w}\big)^{e^{-\tau t}} \qquad \text{in } [0,\infty). \end{aligned} \tag{4.47}$$

One can deduce from (4.47) that w' and w'' converge to zero as $t \to \infty$, and w converge to some constant C as $t \to \infty$. Furthermore, direct calculations yield the value of C for the following three cases:

(1) If $\beta - 2N > a\beta$, then $\tau = a\beta - 2$ and (4.47) gives

$$\lim_{t\to\infty} w(t) = -\frac{\lambda e^{aK}}{(a\beta-2)^2}. \tag{4.48}$$

(2) If $\beta - 2N = a\beta$, then $\tau = a\beta - 2$ and (4.47) gives

$$\lim_{t\to\infty} w(t) = -\frac{\lambda e^{aK} + e^K}{(a\beta-2)^2}. \tag{4.49}$$

(3) If $\beta - 2N < a\beta$, then $\tau = \beta - 2(N+1)$ and (4.47) gives

$$\lim_{t\to\infty} w(t) = -\frac{e^K}{(\beta-2N-2)^2}. \tag{4.50}$$

Proposition 1.4 is therefore established by combining (4.46), and (4.48)–(4.50). □

Remark 4.3. The results of Theorem 1.3 and Proposition 1.4 go only to the case where $\lambda > 0$. However, our methods can also be applied to (1.1) with $\lambda = 0$, in which case some similar results can be obtained. Actually, consider any condensate solution u of the semilinear elliptic equation

$$-\Delta u = |x|^{2N} e^u \quad \text{on} \quad \mathbb{R}^2, \tag{4.51}$$

where $N \geq 0$ is arbitrary, in the sense that

$$\beta = \int_{\mathbb{R}^2} |x|^{2N} e^{u(x)} dx < \infty. \tag{4.52}$$

It then follows from [11, 16], and the references therein, that $\beta = 4(N+1)$, and

$$-4(N+1)\ln(|x|+1) - C \leq u(x) \leq -4(N+1)\ln(|x|+1) + C \quad \text{on} \quad \mathbb{R}^2.$$

Further, the above method of this section can be used to establish that there exists a constant $K = K\big(N, u(0)\big) \in \mathbb{R}$ such that

$$u(x) \sim -4(N+1)\ln|x| + K \quad \text{as} \quad |x| \to \infty.$$

Moreover, we have the following refined results:

(1) If $N \notin \mathbb{N}$, then
   (a) either there exist a natural number k_0 satisfying $0 < k_0 < 2(N+1)$, $A \in \mathbb{R}$ and $\theta_0 \in S^1$ with $A \neq 0$, such that

$$u(x) \sim -4(N+1)\ln|x| + K + A\sin(k_0\theta + \theta_0)|x|^{-k_0} \quad \text{as} \quad |x| \to \infty, \tag{4.53}$$

   (b) or

$$u(x) \sim -4(N+1)\ln|x| + K - \frac{e^K}{(2N+2)^2}|x|^{-2N-2} \quad \text{as} \quad |x| \to \infty. \tag{4.54}$$

(2) If $N \in \mathbb{N}$, then there exist a natural number k_0 satisfying $0 < k_0 < 2(N+1)$, and $A \in \mathbb{R}$ and $\theta_0 \in S^1$ such that

$$u(x) \sim -4(N+1)\ln|x| + K + A\sin(k_0\theta + \theta_0)|x|^{-k_0} \quad \text{as} \quad |x| \to \infty. \tag{4.55}$$

Furthermore, if (4.55) holds for k_0 satisfying $1 \leq k_0 \leq 2N$, then it necessarily has $A \neq 0$ in (4.55).

(3) If $u = u(r)$ is a radial condensate solution of (4.51), then there exists a constant $K = K\big(N, u(0)\big) \in \mathbb{R}$ such that we have the asymptotic behavior

$$u(r) \sim -4(N+1)\ln r + K - \frac{e^K}{(2N+2)^2} r^{-2N-2} \quad \text{as} \quad r = |x| \to \infty. \tag{4.56}$$

One can note that the above asymptotic results of (4.53)–(4.56) are well matched with those explicit condensate solutions discussed in [10, 11, 16].

5. Symmetry under Inversions when $a = \frac{1}{N+1}$

In this section, we apply the "shrinking-sphere" method, c.f. [16], to establishing Theorem 1.5, which gives the "symmetry under the inversions", $i.e.$, the invariance of (1.1) with respect to "inversions", for the special case $a = \frac{1}{N+1}$.

Recall that any condensate solution $u = u(x)$ of (1.1) satisfying (1.7) is smooth at the origin, and u has the decay rate $\beta = 4(N+1)$ when $a = \frac{1}{N+1}$. We start with the following lemma by applying Theorem 1.3.

Lemma 5.1. *For $a = \frac{1}{N+1}$, suppose $u(x)$ is a condensate solution of (1.1) satisfying (1.7), and let $K = K\big(\lambda, a, N, \beta, u(0)\big)$ be the constant given in Theorem 1.3. Set*

$$v(x) = u\big(\frac{x}{|x|^2}\big) + 4(N+1)\ln\frac{1}{|x|}, \quad \textit{where} \quad x \in \mathbb{R}^2. \tag{5.1}$$

Then v is also a condensate solution of (1.1), and

$$K = v(0) = \lim_{|x|\to\infty} \Big(u(x) + 4(N+1)\ln|x|\Big). \tag{5.2}$$

Proof. In view of (5.1), a simple change of the coordinates gives

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \big[\lambda e^{\frac{v(x)}{N+1}} + |x|^{2N} e^{v(x)}\big] dx = 4(N+1) < \infty.$$

Direct calculations show that $v(x)$ satisfies (1.1) in $\mathbb{R}^2 \setminus \{0\}$. Since Theorem 1.3 and (5.1) imply that $K = v(0) = \lim_{|x|\to\infty}\big(u(x) + 4(N+1)\ln|x|\big)$, v admits a removable singularity at zero. Therefore, v is also a condensate solution of (1.1). □

In order to apply the "shrinking-sphere" type method, for any $\mu > 0$ we now introduce the function

$$u_\mu(x) = u(\mu x) + 4(N+1)\ln\mu,$$

and define the domain

$$E_\mu = \big\{x \in \mathbb{R}^2 : |x| > \frac{1}{\sqrt{\mu}}\big\}.$$

It is then easy to check that

$$\begin{aligned} -\Delta u_\mu(x) &= \lambda\mu^2 e^{\frac{u(\mu x)}{N+1}} + \mu^{2(N+1)}|x|^{2N} e^{u(\mu x)} \\ &= \lambda\mu^{-2} e^{\frac{u_\mu(x)}{N+1}} + \mu^{-2(N+1)}|x|^{2N} e^{u_\mu(x)} \quad \text{in} \quad E_\mu, \end{aligned}$$

which therefore gives the following lemma.

Lemma 5.2. *Let $a = \frac{1}{N+1}$, and suppose $u(x)$ is a condensate solution of (1.1) satisfying (1.7). Then for any $\mu \geq 1$, we have*

$$-\Delta u_\mu \leq \lambda e^{\frac{u_\mu(x)}{N+1}} + |x|^{2N} e^{u_\mu(x)} \quad \textit{in} \quad E_\mu, \tag{5.3}$$

and the equality holds if and only if $\mu = 1$.

For any $\mu > 0$, if we set

$$v_\mu(x) = u\big(\frac{x}{\mu|x|^2}\big) + 2(N+1)\ln\frac{1}{\mu|x|^2}, \tag{5.4}$$

then for $v(x)$ defined in (5.1), we have

$$v_\mu(x) = v(\mu x) + 2(N+1)\ln\mu.$$

One can check that $v_\mu(x)$ also satisfies (5.3).

Proof of Theorem 1.5. By applying lemma 5.1, without loss of generality, we can assume that $u(0) \geq K = v(0)$, and hence $\tau \geq 1$. Set

$$w_\mu(x) = u(x) - v_\mu(x), \quad \mu > 0,$$

where $v_\mu(x)$ is defined by (5.4). Since $0 \notin E_\mu$, w_μ is smooth in E_μ. Moreover, Lemma 5.2 implies that for any $\mu \geq 1$, w_μ satisfies

$$\begin{cases} -\Delta w_\mu \geq \big(\lambda f_1(x) + f_2(x)\big)w_\mu & \text{in } E_\mu, \\ \quad w_\mu = 0 & \text{on } \partial E_\mu, \end{cases} \tag{5.5}$$

where

$$f_1(x) = \int_0^{\frac{1}{N+1}} e^{tu+(\frac{1}{N+1}-t)v_\mu}dt, \quad f_2(x) = |x|^{2N}\int_0^1 e^{tu+(1-t)v_\mu}dt.$$

Since

$$v_\mu(x) \leq \max_{x\in\mathbb{R}^2} u(x) \quad \text{in } \bar{E}_\mu,$$

it follows that there exist constants $C_1, C_2 > 0$ such that

$$0 \leq \lambda f_1(x) + f_2(x) \leq \lambda C_1 + C_2|x|^{2N} \quad \text{in } \bar{E}_\mu. \tag{5.6}$$

Claim 1. There exists a finite number $\mu_0 = \mu_0(\lambda, N) > 1$ such that for any $\mu > \mu_0$, we have

$$w_\mu(x) > 0 \quad \text{in } E_\mu.$$

Towards the claim, we set

$$b_0 = \min\Big\{1, \frac{1}{\sqrt{\lambda(C_1+C_2)e}}\Big\}, \tag{5.7}$$

where C_1 and C_2 are given by (5.6). For any $\mu > \frac{1}{b_0^2}$, we also set

$$B_\mu = \big\{x \in \mathbb{R}^2 : \frac{1}{\sqrt{\mu}} < |x| < b_0\big\}.$$

Define

$$h_\lambda(x) = e - e^{\sqrt{\lambda(C_1+C_2)e}|x|} > 0, \;\; x \in B_\mu,$$

which satisfies

$$\begin{aligned} &\Delta h_\lambda(x) + \big(\lambda f_1(x) + f_2(x)\big)h_\lambda(x) \\ &\leq -\lambda(C_1+C_2)e^{1+\sqrt{\lambda(C_1+C_2)e}|x|} + \lambda(C_1+C_2|x|^{2N})\big(e - e^{\sqrt{\lambda(C_1+C_2)e}|x|}\big) \\ &< -\lambda(C_1+C_2)e + \lambda(C_1+C_2|x|^{2N})e \leq 0 \quad \text{in } B_\mu, \end{aligned}$$

since $b_0 \leq 1$. It now follows from [6] that for any $\mu > \frac{1}{b_0^2}$, the maximum principle holds for the operator

$$L_{\lambda,\mu} = \Delta + \big(\lambda f_1(x) + f_2(x)\big) \quad \text{in } B_\mu.$$

Since $u(x)+4(N+1)\ln|x|$ is continuous away from the origin, it follows from (5.2) that for b_0 defined in (5.7), there exists a constant $C>0$ such that

$$\big|u(x)+4(N+1)\ln|x|\big|<C \quad \text{for } |x|\geq b_0.$$

This yields that for $|x|\geq b_0$,

$$\begin{aligned} w_\mu(x) &= u(x)+4(N+1)\ln|x|-u\big(\frac{x}{\mu|x|^2}\big)+2(N+1)\ln\mu \\ &\geq -C-\max_{x\in\mathbb{R}^2}u(x)+2(N+1)\ln\mu. \end{aligned}$$

Therefore, there exists a sufficiently large $\mu_0=\mu_0(\lambda,N)>\frac{1}{b_0^2}\geq 1$ such that for any $\mu>\mu_0$, we have

$$w_\mu(x)>0 \quad \text{holds for any } |x|\geq b_0. \tag{5.8}$$

Together with (5.5), this yields that $L_{\lambda,\mu}w_\mu(x)\leq 0$ in B_μ and $w_\mu(x)\geq 0$ on ∂B_μ, from which the maximum principle gives that $w_\mu(x)>0$ in B_μ. Further, it follows that $w_\mu(x)>0$ in E_μ, and Claim 1 is hence established.

We now define the set

$$\Lambda=\big\{\mu\geq 1:\ w_s(x)>0 \text{ in } E_s \text{ for all } s\geq\mu\big\}.$$

It then follows from Claim 1 that the set Λ is non-empty, and for any $\mu\in\Lambda$ we have

$$\begin{aligned} 0\leq \lim_{|x|\to\infty}w_\mu(x) &= \lim_{|x|\to\infty}\Big(u(x)+4(N+1)\ln|x|\Big) \\ &\quad -\lim_{|x|\to\infty}u\big(\frac{x}{\mu|x|^2}\big)+2(N+1)\ln\mu \\ &= K-u(0)+2(N+1)\ln\mu=2(N+1)\ln(\mu/\tau), \end{aligned}$$

where $\tau\geq 1$ is defined as in (1.24). Therefore, if $\mu\in\Lambda$ we then have $\mu\geq\tau$, which gives $\inf\Lambda\geq\tau$. We next prove the following identity.

Claim 2.

$$\tau=\inf\Lambda. \tag{5.9}$$

To prove Claim 2, the key ingredient of our argument is a version of the maximum principle for any domain with smaller measure, which was used and proven in Berestycki and Nirenberg [6] and was attributed there to Varadhan. We argue by contradiction: on the contrary, suppose $m=\inf\Lambda>\tau$. We shall apply the maximum principle to the operator $L_{\lambda,\mu}=\Delta+\big(\lambda f_1(x)+f_2(x)\big)$ on the subdomain of $D=\{x\in\mathbb{R}^2:\ \frac{1}{\sqrt{m}}<|x|<\frac{1}{\sqrt{\tau}}\}$, where $m<\infty$. Indeed, since $\tau\geq 1$ we observe from (5.6) that $\lambda f_1(x)+f_2(x)$ is uniformly bounded on D for any $\tau\leq\mu\leq m$. Thus, by the maximum principle [6], there exists a small constant $\delta=\delta(\lambda,N)>0$, independent of μ, such that for any subdomain $\Omega\subset D$ with $meas(\Omega)<\delta$, the condition

$$L_{\lambda,\mu}w\leq 0 \quad \text{in } \Omega, \qquad \lim_{x\to\partial\Omega}\inf w\geq 0 \quad \text{on } \partial\Omega \tag{5.10}$$

implies $w\geq 0$ on Ω.

We now choose a sufficiently small $0<\varepsilon_0<m-\tau$ such that

$$meas\Big\{x\in\mathbb{R}^2:\ \frac{1}{\sqrt{m}}<|x|<\frac{1}{\sqrt{m-\varepsilon_0}}\Big\}<\delta. \tag{5.11}$$

Let $R>0$ be sufficiently large so that

$$\big|u(x)+4(N+1)\ln|x|-K\big|<(N+1)\ln\big(\frac{m-\varepsilon_0}{\tau}\big) \quad \text{for } |x|\geq R, \tag{5.12}$$

and

$$\big|u(y)-u(0)\big|<(N+1)\ln\big(\frac{m-\varepsilon_0}{\tau}\big)\quad \text{for } |y|\le\frac{1}{R}. \tag{5.13}$$

Hence, for $\mu\ge m-\varepsilon_0>\tau\ge1$, it follows from (5.12) and (5.13) that

$$\begin{aligned} w_\mu(x) &= \big(u(x)+4(N+1)\ln|x|-K\big)+\big[u(0)-u\big(\frac{x}{\mu|x|^2}\big)\big]+2(N+1)\ln\frac{\mu}{\tau}\\ &> -2(N+1)\ln\big(\frac{m-\varepsilon_0}{\tau}\big)+2(N+1)\ln\frac{\mu}{\tau}\ge 0\quad \text{for } |x|\ge R. \end{aligned} \tag{5.14}$$

On the other hand, the definition of m and the Hopf lemma give that for any $\mu\ge m$,

$$w_\mu>0\quad \text{in } E_\mu,\qquad \partial_r w_\mu>0\quad \text{for } r=|x|=\frac{1}{\sqrt{\mu}}. \tag{5.15}$$

When $\mu=m$, in particular, we obtain that for any $\varepsilon\in(0,\varepsilon_0)$, there exists $\alpha=\alpha(\varepsilon)>0$ such that

$$w_m(x)>\alpha\quad \text{for } \frac{1}{\sqrt{m-\varepsilon}}\le|x|\le R.$$

Therefore, the compactness argument gives that there exists a constant $\mu_1\in(m-\varepsilon,m)$ such that for any $\mu_1\le\mu\le m$,

$$w_\mu(x)>0\quad \text{for } \frac{1}{\sqrt{m-\varepsilon}}\le|x|\le R.$$

We now conclude from (5.14) that

$$w_\mu>0\quad \text{for } |x|\ge\frac{1}{\sqrt{m-\varepsilon}}. \tag{5.16}$$

For any $\mu_1\le\mu\le m$, however, it is obvious that the set $\Omega=\{x\in\mathbb{R}^2:\ \frac{1}{\sqrt{\mu}}<|x|<\frac{1}{\sqrt{m-\varepsilon}}\}$ is contained in D, and (5.11) gives that $meas(\Omega)<\delta$. Since $L_{\lambda,\mu}w_\mu\le0$ in Ω and $w_\mu\ge0$ on $\partial\Omega$, the conclusion (5.16) and the maximum principle (5.10) yield that $w_\mu>0$ on Ω for any $\mu_1\le\mu\le m$. Furthermore, we conclude from this fact and (5.16) that $w_\mu>0$ holds in E_μ for any μ satisfying $\mu_1\le\mu\le m$. This contradicts the definition of m, and hence the Claim 2 is proved.

Claim 2 now implies that for $\tau=\inf\Lambda$,

$$\begin{aligned} -\Delta w_\tau &\ge \big(\lambda f_1(x)+f_2(x)\big)w_\tau && \text{in } E_\tau,\\ w_\tau &\ge 0 && \text{in } E_\tau,\\ w_\mu &= 0 && \text{on } \partial E_\mu. \end{aligned}$$

Since $w_\tau\to0$ as $|x|\to\infty$, it follows that $w_\tau\equiv0$ in $\bar{E}_\tau$, or equivalently,

$$u(x)=u\big(\frac{x}{\tau|x|^2}\big)+2(N+1)\ln\frac{1}{\tau|x|^2}\quad \text{for } |x|\ge\frac{1}{\sqrt{\tau}}. \tag{5.17}$$

Note that if $|x|<\frac{1}{\sqrt{\tau}}$ then $y=\frac{x}{\tau|x|^2}$ satisfies $|y|\ge\frac{1}{\sqrt{\tau}}$, and it then follows from (5.17) that

$$u(y)=u\big(\frac{y}{\tau|y|^2}\big)+2(N+1)\ln\frac{1}{\tau|y|^2},$$

which is equivalent to

$$u\big(\frac{x}{\tau|x|^2}\big)=u(x)+2(N+1)\ln(\tau|x|^2).$$

Therefore, (5.17) holds in $\mathbb{R}^2$, and $u(0) = \lim_{|x|\to\infty}\big[u(x) + 4(N+1)\ln|x| + 2(N+1)\ln\tau\big]$.

Finally, (5.15) implies that

$$\partial_r w_\mu > 0 \quad \text{at } r = |x| = \frac{1}{\sqrt{\mu}}$$

for any $\mu \geq \tau$, which in turn implies that

$$r\partial_r u + 2(N+1) > 0 \quad \text{for } 0 < r = |x| < \frac{1}{\sqrt{\tau}}.$$

On the other hand, it easily follows from (5.17) that

$$r\partial_r u + 2(N+1) = 0 \quad \text{for } r = |x| = \frac{1}{\sqrt{\tau}},$$

and

$$r\partial_r u + 2(N+1) < 0 \quad \text{for } r = |x| > \frac{1}{\sqrt{\tau}},$$

from which we obtain (1.25). □

6. Asymptotic Behavior of Noncoinciding Condensate Solutions

The main purpose of this section is to prove Theorem 1.6 on the asymptotic behavior of noncoinciding condensate solutions of (1.4) satisfying (1.26). Since the proof of Theorem 1.6 is similar to that of Theorem 1.2, in this section we only sketch the main idea of the proof. Similar to Lemma 3.1, we begin with the following crucial lemma.

Lemma 6.1. *Suppose u is a solution of (1.4) satisfying*

$$\beta := \frac{1}{2\pi}\int_{\mathbb{R}^2}\big[\lambda e^{au(x)} + \big(\prod_{i=1}^{N}|x-p_i|^2\big)e^{u(x)}\big]dx < \infty. \tag{6.1}$$

Then e^{au} and $(\prod_{i=1}^{N}|x-p_i|^2)e^u$ are bounded in $\mathbb{R}^2$.

Proof. We first prove that e^{au} and $(\prod_{i=1}^{N}|x-p_i|^2)e^u$ are bounded in the exterior domain $B_R^c(0) := \{x \in \mathbb{R}^2 : |x| \geq R\} \subset\subset \mathbb{R}^2$, where $\max\big\{|p_i|, 1 \leq i \leq N\big\} \leq R < \infty$ is fixed.

Consider any fixed point $x_0 \in \mathbb{R}^2$ satisfying $|x_0| \geq R+1$, and let $B_1(x_0)$ be a unit ball centered at x_0. Set

$$f(x) = u(x) + 2N\ln|x_0|, \tag{6.2}$$

such that $f = f(x)$ satisfies the equation

$$-\Delta f = \frac{\prod_{i=1}^{N}|x-p_i|^2}{|x_0|^{2N}}e^f + \frac{\lambda}{|x_0|^{2Na}}e^{af} \quad \text{in } B_R^c(0). \tag{6.3}$$

Note that for above choices of R and x_0, the assumption (6.1) implies

$$\int_{B_R^c(0)}\Big[\frac{\prod_{i=1}^{N}|x-p_i|^2}{|x_0|^{2N}}e^f + \frac{\lambda}{|x_0|^{2Na}}e^{af}\Big]dx < \infty.$$

This gives that for any $0 < \epsilon < \min\big\{1, \frac{1}{a}\big\}$, one can split

$$\frac{\prod_{i=1}^{N}|x-p_i|^2}{|x_0|^{2N}}e^f + \frac{\lambda}{|x_0|^{2Na}}e^{af} = P_1 + P_2 \quad \text{in } B_R^c(0),$$

such that $\|P_1\|_{L^1(B_R^c(0))} < \epsilon$ and $P_2 \in L^\infty(B_R^c(0))$. Following these facts, a similar argument of proving Lemma 3.1 yields that

$$\|f^+\|_{L^\infty(B_{1/4}(x_0))} \le C, \tag{6.4}$$

where C is independent of x_0. Since x_0, with $|x_0| \ge R+1$, is arbitrary, it follows from (6.2) that

$$u(x) \le C - 2N\ln(|x|+1) \quad \text{in} \quad B_R^c(0) := \big\{x \in \mathbb{R}^2 : \, |x| \ge R\big\},$$

which implies that e^{au} and $(\prod_{i=1}^N |x-p_i|^2)e^u$ are bounded in $B_R^c(0)$ for any fixed sufficiently large R.

For above fixed R, we then consider the condensate solution $u(x)$ of (1.4) in $B_{2R}(0)$, instead of considering the solution of (6.3) in $B_R^c(0)$. A similar argument as above then yields that $u(x)$ is bounded in $B_{2R}(0)$. Therefore, e^{au} and $(\prod_{i=1}^N |x-p_i|^2)e^u$ are bounded in $\mathbb{R}^2$. □

Based on Lemma 6.1, similar to Lemma 3.2, we get that any condensate solution u of (1.4) satisfying (6.1) admits the asymptotic behavior

$$-\beta\ln(|x|+1) - C \le u(x) \le -\beta\ln(|x|+1) + C \quad \text{in} \quad \mathbb{R}^2, \tag{6.5}$$

where $\beta > \max\{2(N+1), 2/a\}$. Let (r,θ) be polar coordinates in $\mathbb{R}^2$, we then have

$$\begin{aligned} ru_r &= x_1u_{x_1} + x_2u_{x_2} \\ &= -\beta - \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{y\cdot(x-y)}{|x-y|^2}\Big(\lambda e^{au(y)} + \big(\prod_{i=1}^N |y-p_i|^2\big)e^{u(y)}\Big)dy, \end{aligned}$$

and

$$u_\theta = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\bar{y}\cdot(x-y)}{|x-y|^2}\Big(\lambda e^{au(y)} + \big(\prod_{i=1}^N |y-p_i|^2\big)e^{u(y)}\Big)dy,$$

where $\bar{y} = (y_2, -y_1) \in \mathbb{R}^2$. Moreover, similar to Lemma 3.1 of [11], we further obtain that

$$ru_r \to -\beta \quad \text{and} \quad u_\theta \to 0 \quad \text{as} \quad r \to \infty. \tag{6.6}$$

To complete the proof of Theorem 1.6, the rest is to establish the identity (1.28).

Proof of (1.28) in Theorem 1.6. Multiplying both hand sides of equation (1.4) by $x\cdot\nabla u$ and integrating by parts over the ball $B_K(0)$, one gets that

$$\begin{aligned} &\int_{\partial B_K(0)} r\Big(\frac{1}{2}|\nabla u|^2 - u_r^2\Big)\, ds \\ &= -\int_{B_K(0)} \big(x\cdot\nabla(\prod_{i=1}^N |x-p_i|^2)\big)e^u\, dx - 2\int_{B_K(0)} (\prod_{i=1}^N |x-p_i|^2)e^u\, dx \\ &\quad + \int_{\partial B_K(0)} r(\prod_{i=1}^N |x-p_i|^2)e^u\, ds - 2\int_{B_K(0)} \frac{\lambda}{a}e^{au}\, dx + \int_{\partial B_K(0)} r\frac{\lambda}{a}e^{au}\, ds \\ &= -\int_{B_K(0)} \Big[\big(x\cdot\nabla(\prod_{i=1}^N |x-p_i|^2)\big)e^u + \Big(\frac{2}{a}-2\Big)\lambda e^{au}\Big]\, dx \\ &\quad - 2\int_{B_K(0)} \Big[(\prod_{i=1}^N |x-p_i|^2)e^u + \lambda e^{au}\Big]\, dx + \int_{\partial B_K(0)} r\Big((\prod_{i=1}^N |x-p_i|^2)e^u + \frac{\lambda}{a}e^{au}\Big)\, ds, \end{aligned}$$

where $|\nabla u|^2 = u_r^2 + \frac{1}{r^2}u_\theta$. Note from (6.6) that

$$\int_{\partial B_K(0)} r\Big(\frac{1}{2}|\nabla u|^2 - u_r^2\Big)\, ds \to -\pi\beta^2 \quad \text{as} \quad K \to \infty.$$

On the other hand, it yields from (6.5) that

$$\int_{\partial B_K(0)} r\Big((\prod_{i=1}^N |x-p_i|^2)e^u + \frac{\lambda}{a}e^{au}\Big)\, ds \to 0 \quad \text{as} \quad K \to \infty.$$

Therefore, we conclude that

$$\pi\beta^2 = \int \Big[\big(x\cdot\nabla(\prod_{i=1}^N |x-p_i|^2)\big)e^u + \Big(\frac{2}{a}-2\Big)\lambda e^{au}\Big]\, dx + 4\pi\beta,$$

which implies (1.28). □

Based on the asymptotic estimate (1.27), we finally remark that any condensate solution of (1.4) satisfying (1.26) also admits the refined asymptotic behavior similar to Theorem 1.3. Indeed, similar to Section 4, for $r_0 := 2\max\{|p_i|,\ 1 \le i \le N\}$ we first set

$$v(t,\theta) := u(r,\theta) + \beta\ln r = u(x) + \beta\ln|x|, \quad t = \ln r \ \text{ and } \ r = |x|, \tag{6.7}$$

where β is as in (1.26) and $(r,\theta) \in (r_0,\infty)\times S^1$ are polar coordinates in $\mathbb{R}^2 \setminus B_{r_0}(0)$, to consider an evolution elliptic equation of the form

$$-v_{tt} - v_{\theta\theta} = \lambda e^{(2-a\beta)t}e^{av} + p(t)e^v, \quad (t,\theta) \in (\ln r_0,\infty)\times S^1, \tag{6.8}$$

where $0 \le p(t) \le 2^N e^{(2N+2-\beta)t}$. Applying dynamical systems theory, we then deduce as in Theorem 4.2 that there exists a constant $K = K\big(\lambda, a, N, \beta, u(0)\big)$ such that

$$v(t,\theta) = u(x) + \beta\ln|x| \to K \quad \text{as} \quad |x| \to \infty. \tag{6.9}$$

Based on (6.9), we then set a new transformation

$$V(t,\theta) = u(r,\theta) + \beta\ln r - K \quad \text{in} \quad (\ln r_0,\infty)\times S^1, \tag{6.10}$$

so that $\lim_{t\to+\infty} V(t,\cdot) = 0$, and $V(t,\theta)$ is a uniformly bounded solution of the following evolution elliptic equation

$$-V_{tt} - V_{\theta\theta} = \lambda e^{aK+(2-a\beta)t}e^{aV} + P(t)e^V, \quad (t,\theta) \in (\ln r_0,\infty)\times S^1, \tag{6.11}$$

where $P(t)$ satisfies

$$0 \le P(t) \le 2^N e^{K+(2N+2-\beta)t}, \quad 2N+2-\beta < 0.$$

By a delicate study of Fourier coefficients of $V(t,\cdot)$, one can further obtain that any condensate solution of (1.4) satisfying (1.26) also admits the refined asymptotic behavior similar to Theorem 1.3. We leave the details to the interested reader.

Acknowledgements: The authors thank the referee for valuable remarks on Section 4, which led to the results of Theorem 1.3. The authors also thank Peter Polacik for many fruitful discussions and Gabriella Tarantello for helpful discussions on [16]. R.M.C. was supported in part by NSF grant no. DMS-0908663. D.S. was supported in part by NSF grant no. DMS-0707714.

REFERENCES

[1] J. Ambjorn and P. Olesen, *Anti-screening of large magnetic fields by vector bosons*, Phys. Lett. B **214** (1988), 565–569.

[2] J. Ambjorn and P. Olesen, *On electroweak magnetism*, Nucl. Phys. B **315** (1989), 606–614.

[3] J. Ambjorn and P. Olesen, *A magnetic condensate solution of the classical electroweak theory*, Phys. Lett. B **218** (1989), 67–71.

[4] J. Ambjorn and P. Olesen, *A condensate solution of the electroweak theory which interpolates between the broken and the symmetry phase*, Nucl. Phys. B **330** (1990), 193–204.

[5] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. Partial Differential Equations **16** (1991), 1223–1253.

[6] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bull. Soc. Brazil Mat. Nova **22** (1991), 1–37.

[7] D. Chae, *Existence of a semilinear elliptic system with exponential nonlinearities*, Discrete Contin. Dyn. Syst. **18** (2007), 709–718.

[8] D. Chae, *Existence of multistring solutions of the self-gravitating massive* W*-boson*, Lett. Math. Phys. **73** (2005), 123–134.

[9] X. F. Chen, S. Hastings, J. B. McLeod and Y. S. Yang, *A nonlinear elliptic equation arising from gauge field theory and cosmology*, Proc. Roy. Soc. London Ser. A **446** (1994), 453–478.

[10] S. Chanillo and M. Kiessling, *Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and geometry*, Comm. Math. Phys. **160** (1994), 217–238.

[11] W. Chen and C. Li, *Qualitative properties of solutions to some nonlinear elliptic equations in* $\mathbb{R}^2$, Duke Math. J. **71** (1993), no. 2, 427–439.

[12] W. Chen and C. Li, *A* $\sup + \inf$ *inequality near* $R = 0$, Adv. Math. **220** (2009), no. 1, 219–245.

[13] X. Y. Chen, H. Matano and L. Véron, *Anisotropic singularities of solutions of nonlinear elliptic equations in* $\mathbb{R}^2$, J. Funct. Anal. **83** (1989), no. 1, 50–97.

[14] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[15] Y. Li and W.-M. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Comm. Partial Differential Equations **18** (1993), 1043–1054.

[16] J. Prajapat and G. Tarantello, *On a class of elliptic problem in* $\mathbb{R}^2$*: symmetry and uniqueness results*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 967–985.

[17] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571.

[18] Y. S. Yang, *Solitons in field theory and nonlinear analysis*, Springer-Verlag, New York, 2001.

ROBIN MING CHEN
SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: chenm@math.umn.edu

YUJIN GUO
SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: yjguo@math.umn.edu

DANIEL SPIRN
SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: spirn@math.umn.edu