Math 5385 - Spring 2018
Problem Set 6

Submit solutions to **three** of the following problems.

1. Determine whether $f = xy^3 - z^2 + y^5 - z^3$ is in the ideal $I = \langle -x^3 + y, x^2y - z \rangle$.

2. Assume that $\Bbbk$ is an algebraically closed field. Identify $\mathbb{A}^9(\Bbbk)$ with the space of (3×3)-matrices $A = [a_{i,j}]$. Let $\rho \colon \mathbb{A}^9(\Bbbk) \to \mathbb{A}^9(\Bbbk)$ be the rational map defined by

$$A \mapsto A \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} A^{-1}.$$

   (a) Find equations for the smallest affine variety X containing the image of ρ.

   (b) Show that X is the set of all nilpotent (3×3)-matrices.

3. Use the method of Lagrange multipliers to find the point(s) on the surface defined by $x^4 + y^2 + z^2 - 1 = 0$ that are closest to the point $(1, 1, 1)$ in $\mathbb{R}^3$.
   Hint: Proceed as in Example 3 in §2.8.

4. Suppose that $\Bbbk$ is a field and $\varphi \colon \Bbbk[x_1, \dots, x_n] \to \Bbbk[x_1]$ is a ring homomorphism that is the identity on $\Bbbk$ and maps x_1 to x_1. Given an ideal $I \subseteq \Bbbk[x_1, \dots, x_n]$, prove that $\varphi(I) \subseteq \Bbbk[x_1]$ is an ideal.
   Hint: In the proof of Theorem 3.5.2, we use this result when φ is the map that evaluates x_i at a_i for $2 \leq i \leq n$.

5. Consider the ideal $I = \langle x^2y + xz + 1, xy - xz^2 + z - 1 \rangle$ discussed in §3.5.

   (a) Show that the partial solution $(b, c) = (0, 0)$ does not extend to a solution $(a, 0, 0) \in V(I)$.

   (b) In the text, it is shown that $g_o = g_1$ for the partial solution $(1, 1)$. Show that $g_o = g_3$ works for all partial solutions different from $(1, 1)$ and $(0, 0)$.