Name: ____________________

Problem Set 8
Math 4281, Spring 2014
Due: Wednesday, March 26

Quotient rings

1. Prove that if F is a field and $f(x) \in F[x]$ is not irreducible, then $F[x]/\langle f(x)\rangle$ contains zero divisors.

2. Give the addition and multiplication tables of $\mathbb{Z}_2[x]/\langle x^3+x+1\rangle$.

3. Let R and S be commutative rings with 1.
   a. Given an ideal $J \subseteq S$, define $\phi^{-1}(J) := \{a \in R \mid \phi(a) \in J\} \subseteq R$. Prove that this is an ideal in R.
   b. Given an ideal $I \subseteq R$, define $\phi(I) := \{\phi(a) \mid a \in I\} \subseteq S$. Prove that $\phi(I)$ is an ideal in S, provided that ϕ maps onto S.
   c. Given an ideal $I \subseteq R$, show that there is a one-to-one correspondence between $\{$ideals of $R/I\}$ and $\{$ideal of R containing $I\}$.

4. An element a of a commutative ring R with 1 is called *nilpotent* if $a^n = 0$ for some positive integer n.
   a. Find the nilpotent elements in $\mathbb{Z}_8$.
   b. Find the nilpotent elements in $\mathbb{Z}_2[x]/\langle x^3\rangle$.
   c. Show that the collection N of all nilpotent elements in R is an ideal.
   d. Show that the quotient ring R/N has no nonzero nilpotent elements.

Ring isomorphisms

5. a. Prove that the function $\phi\colon \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ defined by $\phi(a+b\sqrt{2}) = a-b\sqrt{2}$ is a ring isomorphism.
   b. Define the function $\phi\colon \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{7})$ by $\phi(a+b\sqrt{3}) = a+b\sqrt{7}$. Is ϕ a ring isomorphism? Is there any isomorphism between these rings?

Throughout the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ____________________