Name: ______________________

Problem Set 6
Math 4281, Spring 2014
Due: Wednesday, March 5

Roots of polynomials

1. Prove that $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$, but $\mathbb{Q}(\sqrt{2}i) \subsetneq \mathbb{Q}(\sqrt{2}, i)$.

2. Find the splitting field for the following polynomials in $\mathbb{Q}[x]$:

   a. $f(x) = x^6 - 1$

   b. $f(x) = x^4 - 10x^2 + 1$ (Hint: Show first that $\pm\sqrt{2} \pm \sqrt{3}$ are the roots.)

3. Suppose that $\alpha \in \mathbb{C}$ is a root of $f(x) \in \mathbb{Q}[x]$. Find the multiplicative inverse of $\beta \in \mathbb{Q}(\alpha)$. (Hint: Use the Euclidean algorithm.)

   a. $f(x) = x^2 + 3x - 3 \in \mathbb{Q}[x],\ \beta = \alpha - 1$

   b. $f(x) = x^3 + x^2 + 2x + 1 \in \mathbb{Q}[x],\ \beta = \alpha^2 + 1$

Irreducible polynomials over the integers

4. List the irreducible polynomials in $\mathbb{Z}_2[x]$ of degrees $2, 3$, and 4.

5. Decide which of the following polynomials are irreducible in $\mathbb{Q}[x]$.

   a. $x^3 + 4x^2 - 3x + 5$

   b. $4x^3 - 6x^2 + 6x - 12$

   c. $x^4 - 180$

   d. $x^4 + x^3 - 6$

6. The polynomial

$$\Phi_n(x) = \frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1$$

is called a *cyclotomic polynomial*. Show that $\Phi_p(x)$ is irreducible over $\mathbb{Q}$ for any prime p. (Hint: Consider $\Phi_p(x + 1)$.)

Throughout the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ______________________