SOME ASPECTS OF FUNCTIONAL ANALYSIS AND ALGEBRA

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1. *Introduction.*

Functional analysis, which has become an independent branch of mathematics at the beginning of this century, occupies one of the central positions in contemporary mathematics. This is explained on the one hand by the fact that functional analysis made use of the main classical methods of analysis and algebra, and on the other hand by the rôle which functional analysis plays in contemporary physical science, especially in quantum physics.

The study of mathematical problems connected with quantum mechanics was a turning point in the development of functional anlysis itself, and at the present time to a great extent it determines the main paths for its development.

It can be said without exaggeration that contemporary functional analysis represents a new and serious step in the development of mathematics.

In the last few years a number of new branches have arisen in functional analysis. Although relatively recently (about 20 to 30 years ago) functional analysis was thought of mainly as the theory of linear normed spaces, at the present time that theory, which is important and, roughly speaking, is completed, cannot even be considered as one of the basic branches of functional analysis.

In general, functional analysis is still far from being completed, but the basic tendencies in its development are considerably clearer now than they were some 15 to 20 years ago.

It is of course impossible to discuss in this paper all the basic questions of functional analysis. Therefore, we will limit ourselves to the consideration of a few selected problems. Although at a first glance the problems considered differ in character, common to all of them is, for instance, the fact that the development of each of these branches is closely connected with and is stimulated by the development of quantum mechanics and the quantum field theory. Perhaps not all the problems which will be discussed lie on the main path of development of functional analysis. However we hope that they will at least aid in determing that path.

2. *Representations of groups.*

One of the main branches of functional analysis in which the cooperation of analytic and algebraic methods typical for that field of mathematics, is clearly seen, is the theory of representations of groups.

The theory of representations developed as a branch of algebra in the study of representations of finite groups. The connection between the theory of representations and analysis was discovered after the study of representations of compact topological groups had begun. From that time on the theory of representations, both in the character of the problems with which it was concerned and in its methods, developed essentially as a branch of functional analysis.

It is sufficient to point out, for instance, the connection of the theory of representations with the almost periodic functions, the spherical functions, or the generalized spherical functions, which arise in the representations of the group of rotations of threedimensional space [1], [2].

The development of the theory of representations of groups was stimulated first by the quantum theory and later by the quantum field theory. In particular, from these theories it became clear that the theory of representations is one of the basic mathematical methods for the study of symmetries (invariancies) occurring in physics.

The analytic character of that theory is, naturally, most clearly seen in the representations of Lie groups. We shall limit ourselves to the best known case of semi-simple Lie groups [5], [7], which includes such important groups as, for example, the group of unimodular matrices, the orthogonal group, the Lorentz group, etc.

An important example of representation is the so-called regular representation. This representation acts in the space of all square integrable functions on a group and consists in making correspond to each element h the linear transformation T_h given by the formula: $T_h\varphi(g) = \varphi(gh)$. Decomposing this representation into irreducible ones, in the case of a compact group one obtains all the irreducible representations of the group under consideration. In the case of semi-simple Lie groups the decomposition of the regular representation gives rise to an important class of irreducible representations, the so-called basic series of irreducible representations. However (and in this respect locally compact groups differ essentially from compact ones), not all the irreducible representations of the semi-simple Lie groups are contained in this class.

The reason for the appearance of the so-called supplementary series may be illustrated by a simple example. Let us consider two similar groups: the group of the rotations of a sphere, i.e. the group of the motions of a surface of constant positive curvature and the group of the motions of a surface of constant negative curvature (Lobachevsky planes).

All irreducible representations of rotation groups are given by spherical functions and can be obtained as follows. Let us consider a Laplace operator on the surface of a sphere (the second differential Beltrami parameter). The eigenfunctions of this operator corresponding to a given eigenvalue (i.e. spherical functions) form the space in which the required irreducible unitary representations act. Let us now consider the Laplace operator in the space of the square integrable functions on a Lobachevsky plane. Its spectrum consists of all numbers from $-\infty$ to 0, and spectral expansion in this case also gives irreducible representations, more explicitely the main series of representations of the group of the motions of the Lobachevsky plane. All the eigenfunctions decrease in this case as $e^{-\sqrt{k}r}$, where $-k$ is the curvature of the Lobachevsky plane. However, if we should only require that the eigenfunctions considered be bounded, then more eigenfunctions might arise, corresponding to the supplementary part of the spectrum — the segment from 0 to k. Such a difference between the case of square integrability and the case of boundedness is connected with the fact that the circumference of a circle on a Lobachevsky plane increases with the radius r as $e^{-\sqrt{k}r}$, and, therefore, the class of square integrable functions is much narrower than the class of bounded functions. These supplementary eigenfunctions do not occur in the expansion of square integrable functions, and give rise to a supplementary series of irreducible representations. More exactly, eigenfunctions depending only on the radius are positive definite functions, and with their aid supplementary series of representations may be realized. This example shows that for locally compact homogeneous spaces, in contrast to compact ones, the collection of those invariant elementary positive definite functions, which are required for the Plancherel expansion theorem (the expansion of square integrable functions into an integral analogous to the Fourier integral) is essentially different from the collection of all invariant elementary positive definite functions.

Apparently, supplementary series of representations appear in those cases where there is such a neighbourhood U of the unity in the group, that the measure of U^n increases as a geometric progression.

Before exposing the results related to the representations of the basic series it is desirable to make a few remarks on the so-called dimensions of functional spaces. As is well known, spaces of square integrable functions of any number of variables are isomorphic. However, as far as we know, in concrete questions of analysis it never happens that the space of „all" the functions of a certain number of variables is effectively transformed one-to-one into the space of „all" the functions of another number of variables. It may be said without exaggeration that for analysis the isomorphism of all Hilbert spaces has no more importance than, let us say, for algebraic geometry the fact that a curve and a surface are sets of equal power.

We shall attempt to determine in a given space of functions the number of variables occurring in the functions of the space where the representations of the semi-simple Lie groups are defined. It is possible here to make a comparison with fundamental Burnside theorems on finite groups, according to which the number of nonequivalent representations of a finite group G is equal to the number of classes of conjugated elements in a given group and the sum of the squares of the dimensions of all (nonequivalent) representations is equal to the order of the group. The proof of these theorems is based on the decomposition of regular representations into irreducible ones. Similar facts hold for representations of the Lie groups if the dimensions of functional spaces are conceived in the corresponding way. We shall illustrate this taking as an example the group of complex unimodular matrices with determinant equal to 1.

An element of this group is determined by the values of $n^2 - 1$ complex parameters. A class of conjugated elements is in general determined by the values of $n - 1$ parameters (it consists of all the matrices with a given set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1 \lambda_2 \ldots \lambda_n = 1$). The analogue of the first Burnside theorem learns that the representations of the groups mentioned above are determined by $n - 1$ complex (i.e. $2n - 2$ real) parameters. And the representations of the basic series are, in fact, determined by the values of $2n-2$ parameters ($n - 1$ integral ones and $n - 1$ real ones).

Let us determine now in a functional space the number of variables occurring in the functions of the space on which the irreducible representations are defined. The regular representation acts in the space of the functions on the group, i.e. in the space of functions of $n^2 - 1$ complex parameters; it is decomposed into irreducible representations. Let l represent the dimension of the functional space on which an irreducible representation of a given group acts. The analogue of Burnside's second theorem in this case will be expressed by the relationship $n^2 - 1 = n - 1 + 2l$, from which it follows that $l = \frac{n(n-1)}{2}$, i.e. that each of the irreducible representations is realized in a space of functions depending on $\frac{n(n-1)}{2}$ parameters. This is what really occurs. It is highly improbable that there exists a realization of these representations in a space of „all" the functions depending on a different number of variables, which in some way is natural. It would be interesting to develop a theory in which these considerations would become exact.

Let us now return to the description of the irreducible representations of semi-simple Lie groups. We shall limit ourselves to the case of the group of complex unimodular matrices, as the picture in that case is typical for the general case. The representation of the group of second order matrices operates in the space of functions of a complex variable z subject to fractional linear transformations $\frac{\alpha z+\beta}{\gamma z+\delta}$ given by the second order matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. In constructing the representations of the group of matrices of the nth order it is important, in the first place, to define correctly the conception of fractional linear transformation for this case. The conventional definition of fractional linear transformations is unsatisfactory, since it would lead to representations in the space of functions of $n-1$ variables and not in the space of functions of $\frac{n(n-1)}{2}$ variables, as should occur in accordance with what has been said above. The generalization of the fractional linear transformations necessary in this case is as follows: let us take a point in the n-dimensional projective space, a straight line passing through this point, a plane passing through this straight line, etc. We shall call such a combination a *generalized linear element*. It is not difficult to check that in the $(n-1)$-dimensional projective space the generalized linear element is determined by the values of $\frac{n(n-1)}{2}$ parameters. The representations of the group under consideration are obtained as follows: a measure is introduced in some way in the space of the linear elements, and in the space of the functions of "z" which are square integrable with this measure the operator T_g corresponding to the element "g" is determined by means of the formula: $T_g f(z) = f(zg)\,\alpha(zg)$, where "$zg$" is the image of the linear element "z" in the projective transformation with the matrix "g", and $\alpha(zg)$ is a fixed function, specific for each representation. The function $\alpha(zg)$ is determined from the requirement that $Tg_1\, Tg_2$ should be equal to $Tg_1 g_2$ and the requirement that the representation should be unitary. The basic series differ from the supplementary one in the manner of choosing the scalar product.

It should be noted that the function α is determined apart from equivalences by the character of the group of diagonal matrices, and for an arbitrary semi-simple group, by the character of a similar (Cartan) subgroup. This character itself is a generalization of Cartan's highest weight for the case of infinite-dimensional representation.

Considering the space of functions dependent on linear elements "with omissions" (for example now a point, then a plane, etc.) we obtain a degenerated series of representations. Let us assume $z_{n_1 n_2 \ldots n_k}$ to be the respective "incomplete" linear element, $n_1, n_2, \ldots, n_k$ being the dimensions of the manifolds comprising the given linear element. Each set of dimensions has as degenerated series of its own.

The spaces of linear elements "z" and $z_{n_1 n_2 \ldots n_k}$, as well as the respective representation series may be constructed in a similar way for any semi-simple Lie group. It is possible to establish which of these representations are equivalent.

Continuing to develop the procedures expounded in [5], [6] and in [70], M. A. Naimark proved that the representations described here are all representations of classical complex groups.

The problem of the equivalence of representations can be solved using the classical theory of characters which, and this was a surprise at the time of discovery, is fully applicable to the case of infinite-dimensional representations. It is interesting to note that the formulae of the theory of characters are, in this case, not at all more complicated and, in some cases, are simpler than the formulae of finite-dimensional representations. The existence of characters for complex semi-simple groups was proved by direct calculation on the basis of the apparent type of representation for these groups which was already known at that time [5]. Harish Chandra [7] and Godement [13] proved remarkable theorems about the existence of characters, for any semi-simple group, complex or not.

Interesting new questions arise in studying the representations of real semi-simple Lie groups. When considering, for example, any of the real forms of the group of complex unimodular nth order matrices, the same calculation of the number of parameters given above shows that irreducible unitary representations should be realized in a space of functions depending on $\frac{n(n-1)}{2}$ real parameters $\left(\text{i.e. in a functional space of real dimension } \frac{n(n-1)}{2}\right)$. In order to obtain actually these representations, let us again consider the space "Z" of linear elements which correspond to a complex group. The space "Z" decomposes into parts, which are transitive with respect to the real form. One of these parts is a manifold of real dimension $\frac{n(n-1)}{2}$. On this manifold a functional space is constructed, in which representations of the real group are assigned in a fashion similar to the one described above for a complex group.

For the transitive components whose real dimension is $\frac{n(n-1)}{2} + r$, where $r > 0$, the representation is realized in a space of functions which are arbitrary functions of $\frac{n(n-1)}{2} - r$ real parameters and depend analytically on r complex parameters. The transitive manifold of the highest possible dimension $\left(\frac{n(n-1)}{2}\right.$ complex variables, i.e. $n(n-1)$ real variables$\left.\right)$ is of special interest. The corresponding irreducible representations of the real group are constructed in a space of functions which are analytic in all $\frac{n(n-1)}{2}$ parameters. A similar construction may be used for degenerated series as well. In doing so one should consider the decomposition of the respective spaces of "incomplete linear elements" into transitive components. In this manner we can describe, for example, the representations of the real unimodular group [11] previously described by Bargman [1] for $n = 2$.

The analyticity with respect to some of the variables becomes manifest in the theory of representations in the following manner. Every transitive homogeneous space is a space of the classes of conjugate elements with respect to some subgroup. Functions on a homogeneous space may be considered as functions on a group G, which are constant on classes of conjugate elements with respect to some subgroup K. In passing from the group G to its real form the intersection of this real form with the subgroup K or with a conjugate of it may be partially imaginary. The requirement of constancy on the classes of conjugate elements is replaced by the requirement of analyticity, just as constancy on straight lines parallel to the straight line $x = y$ $\left(\text{i.e. the fulfilment of the equation } \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\right)$ becomes analyticity of f $\left(\text{i.e. the fulfilment of the Cauchy-Riemann conditions } \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}\right)$, when that line is replaced by the straight line $x = iy$. The regular way of determining for which parameters analyticity is required is to write the condition that the functions be constant on the classes of conjugate elements, in the complex case, as $X_j f = 0$, where f is a function on the group, and X_j are Lie operators corresponding to infinitesimal transformations in the subgroup. By introducing the parameters in the required way we find that f is independent of some of them and satisfies the Cauchy-Riemann condition for pairs of others. Of special interest in connection with the theory of automorphic functions are the transitive manifolds of the highest dimensions already mentioned, on which the functions are analytic in all parameters. General automorphic functions should be defined as functions on these manifolds, invariant with respect to some discrete subgroup of the corresponding semi-simple group. The study of such automorphic functions would probably be of interest.

For the special case of a unimodular matrix group of the second order a somewhat different connection with the theory of automorphic functions was discovered by S. V. Fomin and the author of this report in a paper [12] on the theory of dynamic systems. Automorphic functions are obtained in this paper by decomposing into irreducible components the representation acting in the space of the functions on G, constant on the classes of conjugate elements with respect to a certain discrete subgroup. In each of the irreducible representations of the discrete series contained in this representation there is a certain automorphic form determining uniquely the given irreducible representation. The full decomposition of the representation of the group of second order matrices mentioned above into irreducible ones, as well as the analogous decomposition for the case when the representation acts in a space of cosets of the discrete subgroup of the semi-simple group G, would without doubt be of interest.

3. *Generalized functions.*

In one way or another generalized functions have been considered in mathematics and its applications for rather a long time.

A substantial contribution to the formation of the concept of generalized functions was made in the works of Hadamard [15], and later of M. Riesz [16], on finite parts of divergent integrals. Generalized functions (Dirac's δ-function, etc.) were systematically used in quantum mechanics beginning with the nineteen twenties. The general concept of generalized functions as functionals was developed by S. L. Sobolev in connection with his investigations on equations of the hyperbolic type. The treatment of generalized functions as linear functionals on some space of sufficiently smooth functions is the most convenient and natural one.

Much was done in developing the theory of generalized functions by L. Schwartz [18] who combined and systematized the material at hand and presented it from a unified standpoint. The appearance of his book fostered the penetration of these concepts into various branches of mathematics. Schwartz has also introduced, with the aid of generalized functions, the notion of the Fourier transform of functions which do not grow faster than a certain power.

It is, however, necessary to generalize the definition of the Fourier transform for a broader class of functions. This can be seen from the following example which is traditional for the Fourier method. In considering the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, as was shown by A. N. Tichonov, the class of functions, which do not increase faster than $e^{|x|^2}$, is the natural class of functions in which the existence and uniqueness of the solution of the Cauchy problem is ensured. However, the various generalizations of the definition of the Fourier integral, given by Bochner [20], Carleman [19] and Schwartz [18], allow the Cauchy problem to be solved only for the class of functions which do not increase faster than a certain power of x. A further generalization of the Fourier transform requires a further development of the idea of generalized function [60], which, incidentally, is important from other points of view as well. Starting with Schwartz's definition we shall generalize the definition of Fourier transform so that it shall fit some classes of fast growing functions [60]. It is important that in order to define the Fourier transform we must, as a rule, use analytic functions.

By the term generalized functions we shall understand functionals over some topological space S of infinitely differentiable functions which approach zero faster than any power of x. We shall call this space the basic space. The Fourier transforms of all the functions of the basic space also form a basic space $\tilde{S}$, to which we shall refer as the dual of S.

If we are given the generalized function T, i.e. the functional $T(\varphi)$ over some basic space S, then in generalizing Schwartz's definition we shall introduce the Fourier transform $\tilde{T}$ as a functional over $\tilde{S}$ determined by the equation $\tilde{T}(\tilde{\varphi}) = T(\varphi)$ $(\varphi \in S, \tilde{\varphi} \in \tilde{S})$. The space Z_p^p of all entire analytic functions $\varphi(z)$ of order of growth not higher than p and of order of decrease along the real axis not lower than p: $|\varphi(z)| \leqq C_1 e^{c_2|z|^p}$, $|\varphi(x)| < C_3 e^{-c_4|x|^p}$ may serve as an example of a basic space. It is possible to mention a number of other spaces, for example the space K of all finite infinitely differentiable functions introduced by Schwartz, and its dual space Z^1 of entire analytic functions $\varphi(z)$ of order of growth not higher than 1, decreasing along each straight line parallel to the axis of x, faster than any power of x:

$$\sup_{-\infty < x < \infty} |x^n \varphi(x + iy)| \leqq C_n e^{c_n|y|} \quad (n = 1, 2, \ldots).$$

For applying the Fourier transform method to a problem (for example to a given type of differential equations) it is important to select the basic space (and, consequently, the set of generalized functions). For instance, if we have a system of differential equations of order p,

$$\frac{\partial u_i}{\partial t} = \sum P_{ik}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) u_k,$$

where the P_{ik} represent polynomials of degree not higher than p, then by introducing the basic space of analytic functions Z_q^q $(p < q < p + \varepsilon)$ we can, with the help of the Fourier transform, prove the uniqueness of the solution of the Cauchy problem for the class of functions which do not grow faster than $e^{|z|^{p'-\varepsilon}}$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. One of the principal points in the proof of this theorem, the fact that $\tilde{Z}_p^p = Z_{p'}^{p'}$, is based on the Phragmén-Lindelöf theorem. Thus, the use of a complex variable plays an important part in this case and this is evidently connected with the very essence of the problem. And indeed from the general theorem formulated here follows the uniqueness of the solution of the Schroedinger equation $\frac{\partial u}{\partial t} = i \triangle u$ in the class of functions which do not grow faster than $e^{|z|^{2-\varepsilon}}$. While for the equation of heat transfer the proof of uniqueness in the same class of functions may be obtained in numerous ways, it is hard to conceive how one could prove this theorem for the Schroedinger equation without using the theory of functions of a complex variable.

Lack of time does not permit us to discuss other applications of generalized functions. We shall only emphasize once more the following two points:

1) In order to be able to use the method of generalized functions it is important that for each problem, or class of problems, one can construct the corresponding basic space. A "universal" basic space valid for all problems does not exist, and it is senseless to attempt to construct it.

2) For a broad class of problems in which generalized functions are used it is necessary to pass to the complex area. This implies the consideration of a basic space consisting of analytic functions (or a space consisting of the solutions of some differential equations).

Another significant aspect of the theory of generalized functions should be mentioned, viz. the calculation of divergent integrals and series. The calculation of divergent integrals is carried out by Hadamard, essentially by dropping the divergent part (or, as a physicist would put it, by the "regularization" of the given divergent integral). M. Riesz obtains the convergence of some types of integrals by applying the method of analytic continuation. It must be underlined that we are speaking here not of summing oscillating integrals, but of giving finite values to integrals which really tend to $+\infty$.

In connection with this group of questions the following two problems are of interest:

I. Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial. Consider the area in which $P > 0$. Let $\varphi(x_1, \ldots, x_n)$ be an infinitely many differentiable function equal to zero outside a certain finite area. We shall examine the functional

$$(P^\lambda \cdot \varphi) = \int\limits_{P>0} P^\lambda(x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n)dx_1 \ldots dx_n.$$

It is necessary to prove that this is a meromorphic function of λ (it would be natural to call it a ζ-function of the given polynomial), whose poles are located in points forming several arithmetical progressions, as well as to calculate the residues of this function.

II. The second problem may be stated as follows: What should one understand by a rational function (as a generalized function) and how might one find its Fourier transform? We shall consider a rational function as a functional on Z^1, i.e. we shall understand the value of this functional to be an integral along some surface parallel at infinity to the real subspace of Z^1 (different surfaces produce different functionals). The Fourier transform of such a functional is the functional on K (a space of finite functions), dual with respect to Z^1. The general case of these Fourier transforms has been studied little. For the special case where the denominator of the rational function is a polynomial of the so called hyperbolic type, they were investigated, in connection with hyperbolic differential equations, by Herglotz [22], Petrovski [69], Garding [23] and Leray [24]. In connection with elliptic equations Fourier transforms of rational functions with "elliptic" denominators were studied by Shapiro [25], Lopatinski [26] and Bochner [27].

4. *Generalized Functions and Representations of Groups.*

The development of the theory of representations of Lie groups in the last few years shows that here, too, generalized functions are a convenient and useful apparatus. The use of generalized functions has, in particular, been proved useful in the investigation of the equivalence and irreducibility of representations (Brouchat [21], Mackey [30] and Mautner [37]). Generalized functions may also be employed conveniently in assigning scalar products which fit the supplementary series (see [6], § 18).

One of the specific questions of the theory of representations, where generalized functions can be used conveniently, is the so-called Plancherel theorem which gives the expansion of the function $f(g)$ on the Lie group into the analogue of the Fourier integral [9, 10]. This is equivalent to the calculation of $f(e)$ (e is the unit of the group), if for each character $\chi(g)$ of this group $\int f(g)\chi(g)dg$ is known.

For classical Lie groups, from the values of this integral, we can easily find the integral of the function $f(g)$, taken in any class of conjugate elements in general position. Thus in the case of compact groups $f(e)$ is determined. For instance, for the group of unitary matrices the conjugate element class is the set of all matrices with given eigenvalues. When all these eigenvalues tend to 1, the respective classes of conjugate elements tend to the unit, just like concentric spheres $x^2 + y^2 + z^2 = c$ shrink into their centre when $c \to 0$.

In the case of a noncompact group, for instance a group of all nth order matrices with determinant 1, the class of conjugate elements in general position is again the set of all matrices with given unequal eigenvalues. In this case, however, when the eigenvalues approach 1, the corresponding class does not reduce to a single matrix at all, but approaches the set of all matrices with the only eigenvalue 1, just as hyperboloids $x^2 + y^2 - z^2 = c$, when $c \to 0$, do not reduce to the origin, but become a cone. The unit matrix itself constitutes a rather complicated singular point in the manifold of all matrices, which have all eigenvalues equal to 1, and it is, therefore, far from obvious how one can find $f(e)$ if one knows the integral of the function on the class of conjugate elements. This problem may be solved with the aid of generalized functions by applying the method due to M. Riesz [9, 10].

In order to explain the essential idea of the method, we shall illustrate it by a simple model problem. Let f be a finite, sufficiently smooth function. Let I_c denote the integral of the function f on the hyperboloid $x^2 + y^2 - z^2 = c$. The problem is to calculate $f(0,0,0)$, if, for each c, I_c is known. Note that for this purpose the integral $\int f(x,y,z)(x^2 + y^2 - z^2)^\lambda \, dxdydz$ can be calculated if one knows only I_c, as on each hyperboloid the second factor is constant. Now, if λ approaches $-\frac{3}{2}$, the integral will approach $f(0,0,0)$. Since on the other hand the same integral can be written in terms of I_c, we obtain an expression for $f(0,0,0)$ in terms of I_c. The general problem stated above for groups is solved in a similar way.

The following problem is closely connected with the one examined. Let $P(x,y,z, \ldots)$ be a polynomial and $f(x,y,z, \ldots)$ a finite function. It is necessary to find the values of f in singular points, the integrals of f along singular lines of the surfaces $P = C$, etc., if the integrals of f over all surfaces of constant level of the polynomial P are known.

Let us consider a few problems arising in connection with the application of the theory of generalized functions to representations.

It is well-known from the theory of representations of compact groups that finding the representations of a compact group is equivalent to finding the representations of the centre of the group ring consisting of those functions which are constant on the classes of conjugate elements, i.e. which satisfy the condition $f(g) = f(g_0 g g_0^{-1})$ for all $g, g_0 \in G$. The product (convolution) of such functions is determined by the formula: $(f_1 . f_2)(g) = \int f_1(g g_1^{-1}) f_2(g_1) \, d\mu(g_1)$.

The representation of the centre of the group ring is given by the formula $f(g) \to \int f(g)\chi(g)dg$ where $\chi(g)$ is a character of the group.

As mentioned above, characters for arbitrary classical groups also exist and also determine uniquely the representation. It would be interesting, in this case, too, to construct the centre of the group ring and thus to obtain characters.

The direct determination of the centre is hindered by the fact that any function belonging to the centre is constant on the classes of conjugate elements, and, consequently, as a rule, is not integrable; therefore the direct calculation of the convolution leads to divergent integrals. A more general problem of the same type is the problem of the construction of a general theory of spherical functions. It is known that general spherical functions are connected with the group of the transformations of a manifold in the same manner as the conventional spherical functions are connected with the group of the rotations of a sphere. Instead of a group of transformations of a manifold one may speak of a group and a given stationary subgroup. Then the representations of the group prove to be connected in a natural way with the ring of functions which are constant on two-sided cosets of the group on this subgroup [3].

So far only the case has been studied, where the respective stationary group is compact [31, 13, 70]. In the general case the functions of the ring, as a rule, are not integrable and the calculation of their convolution leads to divergent integrals. It is quite probable that the application of the theory of generalized functions will allow the construction of a theory of spherical functions for noncompact Lie subgroups as well.

For compact Lie groups there are two ways of proving that the representations constructed are all the possible representations of this group. One of them consists in using the theory of characters, the other in using the Cartan theory of highest weights. We have already mentioned the possibility of applying the first of them to local compact Lie groups. It is interesting that the wider use of the second method is also connected with the theory of spherical functions on noncompact stationary subgroups. For instance, in the case when G is a group of matrices with determinant 1 the subgroup Z of triangular matrices with all diagonal elements equal to one is such a subgroup.

It seems to me that, if the theory of generalized functions is used correctly, the main problems of the theory of representations of locally compact Lie groups will be not more complicated than for compact groups.

5. *Differential Operators.*

The theory of differential equations is one of the sources of modern functional analysis. In 1910 H. Weyl published one of the first works on the spectral properties of differential operators. Several related concepts such as the eigenfunction and the eigenvalue came into use at the beginning of the last century (Fourier method). A comprehensive exposition of the main questions in differential operator theory should require at least a separate report. Therefore, we shall occupy ourselves only with a few special questions.

At the present time the main progress in differential operator theory has been achieved in the following topics grouped somewhat arbitrarily:

1) Differential operators and boundary value problems
2) Spectral theory of differential operators
3) Inverse problems.

To the most important topics of the theory of differential equations belong boundary value problems for various types of equations. The connection between these problems and spectral theory is clearly indicated in the works of Friedrichs. In the last few years new important results have been obtained by Vishik [32, 33, 34, 35]. Unfortunately, due to lack of time, we cannot discuss here these works in detail.

We shall now treat more in detail the spectral theory of differential operators. Let us consider first of all selfadjoint differential operators. The existence of a general theorem for the spectral expansion of an arbitrary selfadjoint operator in differential operator theory cannot be considered as a final solution to the problem, as it is desirable to obtain this expansion as an expansion by ordinary or generalized eigenfunctions and not as an abstract resolution of the unity E_λ.

The expansion by eigenfunctions over a finite interval, in the case of ordinary differential equations, has been known for a long time. In 1910 Weyl considered the case of an expansion over an infinite interval for a differential operator of the second order. The spectral expansion has further been extended to the case of ordinary selfadjoint differential operators of higher orders by M. G. Krein [28] and Kodaira [29], and to the case of partial elliptic differential equations by A. J. Powsner [65] who used Carleman's results for this purpose. Mautner [37] presented an interesting study filling the gap between general spectral expansions and expansions by eigenfunctions. In this study restrictions are imposed on an operator in functional space, allowing the spectral expansion to be accomplished by functions. From Mautner's results it is possible to obtain for elliptic operators the expansions by eigenfunctions (see also Garding [30]).

In the field of partial differential operators progress has been made only for elliptic operators. A solution to the problem of a spectral expansion of an arbitrary (e.g. hyperbolic) operator has not been found as yet. Generalized functions do not give any new results for elliptic operators. However, eigenfunctions should be sought for as generalized functions in the case of hyperbolic differential operators.

Let us consider the problem of a spectral expansion of a dynamic system. Let a system of differential equations be given in a certain analytic manifold:

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n),$$

where the X_i are analytic functions. In the manifold there exists an invariant integral (invariant volume).

This system may be compared with the partial differential operator

$$Z = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_n \frac{\partial}{\partial x_n}$$

(which can be called the Lie operator for this system), characterizing an infinitely small displacement along the trajectory. If a Hilbert space consisting of square integrable functions is considered, the operator iL will be a selfadjoint operator in this space having, in accordance with the general theory, a certain spectral expansion. The rather trivial case where the spectrum of this operator is a pure point spectrum has been investigated completely. As far as we know, the continuous spectrum in a spectral expansion has been investigated only for the so-called geodesic streams in manifolds of constant negative curvature [12]. However, even in this case, the investigation of the corresponding eigenfunctions has not yet been completed. Here it is necessary to prove the existence of a complete system of generalized eigenfunctions for any analytic dynamic system. In all known examples of transitive dynamic systems with a continuous spectrum the multiplicity of this spectrum is infinite. At the same time, the eigenfunctions (considered in the usual way and not as generalized functions) are only constant here. It is interesting to determine whether other generalized eigenfunctions (corresponding let us say to the eigenvalue $\lambda = 0$) exist in such systems.

A second problem in the field of spectral expansion of differential operators is as follows. It is well-known that the theory of eigenvalues is equivalent to the study of a pencil of quadratic forms $A + \lambda E$. The more general problem concerning the study of the pencil $A + \lambda B$, where $B(u,u)$ is a positive definite quadratic form, in the general theory of linear operators (of finite dimensions or not), is equivalent to this one, as we have only to choose a new scalar product. However, in the case of differential operators, where the theorem for the spectral expansion of the unit operator is not the final result of the whole theory but rather a leading thread, the problem for the quadratic differential form $A(u,u) + \lambda\, B(u,u)$ is a new one. An interesting example of such a study is found in the publications of Sobolev [38] and Alexandryan [39]. There a problem is considered which is essentially equivalent to the study of quadrati c forms

$$\iint \frac{\partial^2 u}{\partial x^2} u dx dy + \lambda \iint \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dx dy$$

for functions equal to zero on the boundary of a region G.

An interesting and important problem in the spectral theory of differentia operators is the study of the asymptotic behaviour of the eigenfunctions and eigenvalues. Numerous results have been obtained recently in the study of partial differential equations by B. M. Levitan [41, 42]. The investigation of the asymptotic behaviour of the eigenfunctions and eigenvalues of differential operators is closely related to the so-called quasi-classical approximation in quantum mechanics.

Selfadjoint operators were considered above. Much less progress has been made in the spectral theory of non-selfadjoint operators, although it sphysical application is of great value. In this field we have the early works of Tamarkin and Hille where the case of a finite interval of a straight line is considered and an important work of Carleman where the existence of eigenfunctions for the elliptic differential operator of the second order is proved. The results obtained by M. V. Keldysh [40] in 1951 are an important contribution to this field. He proved that the system of eigenfunctions for an elliptic equation together with the so-called allied functions (the analogue of the basis with respect to which the matrix takes the Jordanform) is complete. In the works of M. V. Keldysh a wide use has been made of the theory of functions of a complex variable and also of his theorems on the general theory of operators (see also Browder [43, 44]).

These results may be considered as the beginning of a new stage in spectral theory, since here, for the first time, a theorem on the completeness of eigenfunctions has been given that goes beyond the field of selfadjoint operators.

A study by M. S. Lifshitz [45] should be mentioned where the possibility of transforming a selfadjoint and completely continuous operator into the triangular form has been proved. M. S. Lifshitz makes use of the methods of the theory of functions of a complex variable and in particular of Potapov's theorem [46], when dealing with the expansion of bounded analytic matric functions in a product, similar to the Blaschke product.

Hardly anything has been done in the spectral theory of non-selfadjoint operators in the case of an infinite region. In this case it is not even clear under what conditions for the operator a spectral expansion is possible. Let us consider the simple example of a differential equation of the first order:

$$Ly = i\frac{dy}{dx} + a(x)y,$$

where $a(x)$ represents a certain complex function. In this example we shall use the following most conservative definition of a spectrum. Let us consider the operator over a finite interval $(-N,N)$ having, for instance, periodic boundary conditions. Then a discrete spectrum $\lambda(N)$ is obtained. Now let $N \to \infty$. The limit of the set $\lambda(N)$ for $N \to \infty$ will be called the spectrum of the operator L along the whole axis. The eigenvalues of the operator L, with periodic boundary conditions, in the interval $(-N, N)$ are equal to

$$\lambda_n^N = \frac{1}{2N}\int_{-N}^{N} a(s)ds + \pi\frac{n}{N}.$$

We can see, for instance, that, if the mean value of the imaginary part of the function $a(x)$ in the interval $(-N, N)$ approaches ∞ as $N \to \infty$, then the whole spectrum goes to infinity as $N \to \infty$. Consequently, in this case (e.g. for the operator $i\frac{dy}{dx} + ix^2y$) there is no spectrum in the finite part of the plane and the very problem of the spectral expansion for such an operator has no meaning.

It is hardly possible to have a spectral expansion over an infinite interval, for example in a Hilbert space for the operator $i\frac{d^2y}{dx^2} + ixy$, which is the "Fourier transform" of the above mentioned operator. As far as we know, the only spectral expansion of non-selfadjoint differential operators in an infinite region is the expansion of the differential operators of the second order shown in the work of Naimark [47].

Finally we shall say something about inverse problems. So-called inverse problems (see also [48, 49]) have become of great importance in connection with certain problems of mathematical physics and especially of quantum mechanics. Let us examine for example Schroedinger's equation

$$-\psi''(r) + V(r)\psi = k^2\psi$$

with the conditions that $\psi(0) = 0$, $\psi'(0) = 1$. Assuming that the potential $V(r) = 0$ for $r \geqq a$ we have, if r is sufficiently large, $\psi(r) = \varrho(k)\sin(kr + \eta(k))$ where $\eta(k)$ is the phase shift. The inverse problem in this case is to find the potential from $\eta(k)$ or $\varrho(k)$. Where there is no discrete spectrum, the potential is determined uniquely by the phase shift, as was proved by Levinson [50]. At about the same time, similar solutions to inverse problems in geophysics were obtained by A. N. Tichonov [51].

The uniqueness of the solution to the inverse problem for an arbitrary

$V(r)$ has been proved by Marchenko [52], who considered, instead of the phase, the so-called spectral function $\varrho(\lambda)$, which in the preceding particular case coincides with the multiplier $\varrho(k)$.

Further topics in the field of inverse problems for equations of the second order are the following questions:

a. What can serve as spectral function and in particular what point set can serve as a spectrum for a differential operator;

b. In which way can the potential be reproduced effectively by $\varrho(k)$?

All these problems were solved by M. G. Krein [53, 54, 55], B. M. Levitan and the author [56]. In these studies the authors derived necessary and sufficient conditions for $\varrho(k)$ being the spectral function of a differential operator. The determination of the coefficients of the equation by means of $\varrho(k)$ was reduced to the solution of a certain Fredholm integral equation [56]. These results make it also possible to penetrate into the nature of the spectrum of differential operators. It follows that:

1) For any closed set in a finite interval there exists a differential operator the spectrum of which in this interval coincides with the given set.

2) Any sequence fulfilling certain asymptotic conditions can be the sequence of eigenvalues for a Sturm-Liouville operator in a finite interval (with corresponding boundary conditions).

The above mentioned integral equation for inverse problems was applied to problems on dispersion theory in quantum mechanics in the works of Jost and Kohn [57]. It was used by Corinalesi [58] in the case of a relativistic particle.

Using the same integral equation M. G. Krein derived important formulas in order to determine $V(r)$ by means of $\varrho(\lambda)$ for a large class of functions $\varrho(\lambda)$, and in particular for all rational functions.

So far we have considered the one-dimensional case of an inverse problem. Little has been done for the case of several independent variables. It would be most natural to consider in this case a problem similar to that of determining $V(r)$ by means of the phase.

Let us consider the equation:

$$-\triangle u + V(x,y,z)u + \varkappa^2 u = 0.$$

Assume that $V=0$ outside a certain finite region and that only equations without any discrete spectrum are to be considered. Let us examine the solution of this equation and also a normal derivative of the solution on a sufficiently large closed surface. The function u and its normal derivative can be divided into two components, u_1 and u_2 say, representing convergent and divergent waves. We may consider these waves (both outgoing and ingoing) without taking into account the derivatives of u_1 and u_2, because the values of the normal derivatives are determined by the values of the function itself. Let $S(k)$ be the operator which to each outgoing wave associates the corresponding ingoing wave. The problem is to determine whether the potential $V(r)$ can be defined by means of the operator $S(k)$. Instead of the operator $S(k)$ the operator $R(\lambda)$ introduced by Wigner can be considered. It establishes a relationship between the function on the surface and its normal derivative [67, 68].

In quantum mechanics this problem can be treated in terms of dispersion theory in the same way as in the one-dimensional case. We mention also an interpretation of this problem in the field of optics. A light wave coming from infinity is dispersed in the points of inhomogeneity defined by a certain function V. We observe a dispersed wave. Is it possible to reproduce the function V using the data, obtained by varying in any manner the oncoming waves and observing the corresponding dispersed waves? If it is possible, how can we do it?

A possible variant of the inverse problem is as follows. Let us examine the external boundary condition problem for the equation

$$-\triangle u + V(x_1,x_2, \dots, x_n)u = \lambda u,$$

with the boundary condition $\frac{\partial u}{\partial n} = 0$. Let us consider the resolution of the unity $E(\lambda)$ corresponding to this operator. $E(\lambda)$ is an integral operator with a kernel $\varrho(P,Q,\lambda)$. If we consider this kernel for points P and Q lying on the boundary of the region, we get the function $\varrho(S_1,S_2,\lambda)$, which is an analogue of the function $\varrho(\lambda)$ in the one-dimensional problem. Now one has to find the potential $V(r)$ knowing $\varrho(S_1,S_2,\lambda)$. Some results for this second problem were obtained by Berezanski [59].

6. *Analysis in Functional Spaces.*

Although the branches of functional analysis mentioned above are comparatively new and are continuing to develop rapidly, they nevertheless have already taken a definite form and acquired, so to say, a "personal" character. This is not true for the group of questions covered by the last section of this report. We shall deal here with problems and procedures which are just beginning to appear; it is, however, possible that in the future they will occupy a central place in functional analysis as a whole. There are a number of physical problems in which, apart from difficulties of physical character, other difficulties arise, due to the absence of a sufficiently general, adequate mathematical apparatus. Some questions of quantum electrodynamics, the theory of turbulence, etc. are of this type. Lately such a mechanism has begun to take shape. It might be called analysis in functional spaces. We shall illustrate the questions and methods arising here on a problem of quantum electrodynamics. We shall not use any data of quantum electrodynamics, but shall only consider the simplest model equation. On the basis of ideas which are to be found in the well-known paper by V.A. Fok [17], Schwinger [62] developed quantum electrodynamics with the aid of functionals in a space (referred to as the space of external sources). In doing so the equations of quantum electrodynamics become a comparatively complicated system of integral equations and lead to relationships between functionals and their variational derivatives. This system of equations could be simplified much and reduced to a linear differential equation in the functional space, or to a system of such equations [66]. We shall demonstrate this procedure by using a very simple equation, the so-called Thiring quantum equation

$$(-\square + \varkappa^2)\psi = \lambda\psi^3 + I,$$

where

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial t^2}$$

and $\psi(x_1,x_2,x_3,x_4)$, as usual in quantum theory, designate certain operators. Following the method of Schwinger we add certain scalar functions $I(x)$ (called external sources) to this equation and reduce it to a system of ordinary differential equations (not containing operators) in the following way. Let us assume that at the moment $t = -\infty$ the external sources $I(x)$ are absent and let e_0 designate the state of vacuum without external sources. If then external sources are introduced, the state e_0 will alter in some way; and at $t = +\infty$ will change into some state e_0^1. Let us assume that Z is equal to the scalar product of the wave functions e_0 and e_0^1 ($|Z|^2$ indicates the probability that the vacuum has remained a vacuum). Z is a linear functional of the external sources I and satisfies the following differential equation:

$$\lambda \frac{\delta^2 Z}{\delta I(x)^2} + (\square - \varkappa^2)\frac{\delta Z}{\delta I(x)} = I(x)Z. \tag{1}$$

Here $\dfrac{\delta Z}{\delta I(x)}$ is the so-called variational-functional derivative, i.e. the limit of the ratio of the increase of Z and $\int \delta I(x)dx$, where the variation $\delta I(x)$ is concentrated in an infinitesimal neighbourhood of the fixed point x_0.

The equation (1) has a similar form as the Airy equation, which is obtained from it, if Z is considered, not as a functional, but as a function and the variational-functional derivatives are replaced by usual derivatives.

It seems natural to call the solutions of the functional equation (1) Airy functionals. They can be obtained in the following way.

Let us consider in the four-dimensional space of variables x_1,x_2,x_3,x_4 a cube of side L, divide it into cells of side τ and assume that Z depends only on the values of I at the joints of these cells. The equation (1) reduces to a system of partial differential equations

$$\frac{1}{\omega^2}\frac{\partial^2 Z}{\partial I_i^2}+\frac{1}{\omega}\sum_k R_{ik}\frac{\partial Z}{\partial I_k}=I_i Z,$$

where ω is the volume of a cell and R^{ik} is the matrix obtained by replacing the operator $\square - \varkappa^2$ by the corresponding difference operator.

We shall solve this equation with the aid of a Laplace transform, i.e. we shall search a solution of the form

$$Z(I_1, I_2, \ldots, I_n)=\int \xi(s_1, \ldots, s_n) e^{i\Sigma I_k s_k} ds_1 \ldots ds_n, \tag{2}$$

where the integration is extended over the surface obtained as the direct product of n contours C_k chosen in the planes of the corresponding complex variables s_k. Thus we obtain the following expression for $\xi(s_1, s_2, \ldots, s_n)$:

$$\xi(s^1, \ldots, s_n)=e^{i\Sigma\frac{s_k^3}{3}+\Sigma R_{ik}s_i s_k}.$$

In order that the integral (2) be different from zero it is necessary that each of the contours C_k extend into infinity. At the same time we must ensure the convergence of the integral (2). This problem can be solved for each of the contours C_k along which one integrates. It is easy to verify that for the convergence of the integral (2) it is necessary and sufficient that each of the contours C_k extend into infinity within one of three angles, I, II and III say. In order to obtain a nonzero integral, one must take the contour C_k in the union of the angles I and II, I and III, or II and III. In the latter case the value of the integral is equal to the sum of its values in the two former cases, so there will be only two independent integrals. In integrating along all the n contours we obtain in this way 2^n independent solutions. Linear combination of the solutions furnishes the general solution of the equation (1). However, if we should now pass to the limit by letting the dimensions of the cells tend to zero ($\tau \to 0$), i.e. if we go over from indices $1, 2, \ldots, n$ to a continuous variable x, then in the limit we shall obtain only two solutions (if in different planes we take the contours in different pairs of angles, then there will be no solution in the limiting case).

If the functional Z has been found, the calculation of various quantum-mechanical effects reduces to the determination of some functional derivatives of Z.

This treatment of quantum-mechanical problems is closely connected with the theory constructed by Feynman [63] on the basis of other considerations. Similar methods are also used by Edwards and Peierls [64].

For a rigorous justification of the method of solution given above it is necessary to be able to answer a number of other questions. For instance, one must justify the transition to the limit for $n \to \infty$. It would be still better to obtain the solution directly by integrating in the functional space. It is desirable to find the asymptotic behaviour of the Airy functionals (in a similar way as was done for the Airy functions). All these specific mathematical questions are of unquestionable interest not only for quantum electrodynamics, where they arise, but for many other fields as well. Questions which are very close to those just discussed, arise, for example, in the theory of turbulence in the interpretation given in a recent work by E. Hopf [61]. Let $u_\alpha(x)$ be the velocity field of a certain liquid. Let us introduce the functions $y^\alpha(x)$ and assume that $Z = \langle e^{\int y^\alpha u_\alpha dx} \rangle$ (here the simbol $\langle\ \rangle$ indicates taking the mean). Zrepresents a functional of $y(x)$. On the basis of the equations of hydrodynamics E. Hopf has given a linear equation for Z in functional derivatives. The similarity with quantum electrodynamics found here is of interest. For instance, the correlation between velocities in different points is calculated, as quantum-mechanical effects, by means of functional derivatives of Z. Here questions of the kind of integration in functional spaces may also play an essential rôle.

It should be noted that some of the concepts of analysis in functional spaces do exist already rather a long time (f.i. N. Wiener's concept of measure in a functional space). However, in the near future, they may occupy a considerably more significant place than they have so far.

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