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6-0

The Bernstein filtration on $A_n(k)$.

- Given a finitely generated left $A_n(k)$-module, we are going to define a dimension $d(M)$. then we will define and study the class of holonomic modules.

$A_n(k) \xrightarrow{\text{Bernstein filtration}} \mathcal{B}$. $\underline{gr^{\mathcal{B}}(A_n(k))}$

associated graded alg.

$\underline{M}$: $\xrightarrow{\text{filtration } \Gamma}$ $gr^{\Gamma}(M)$

graded module over $gr^{B}(A_n(k))$.

$gr^{B}(A_n(k))$ is commutative!, has 2-side ideals!

Apply. Comm. Alg. to define dimension.

Definition:

A graded ring (resp. k-alg.) is a ring (resp. k-alg.) R that admits a decomp.

$$R = \bigoplus_{i \geq 0} R_i$$

← called ith homogeneous component.

as Abelian groups (or k. vector spaces). s.t.

$$R_i \cdot R_j \subset R_{i+j}, \quad \forall i, j \geq 0$$

Polynomials ~~on~~ $R[x_1, \ldots, x_n]$ is of course graded

* but the grade may not be degree!

Let R be a graded ring. M is an R-module. If

$$M = \bigoplus_{i \geq 0} M_i \quad \text{(as Abelian group)} \quad \text{s.t.}$$

$R_i M_j \subset M_{i+j}$, $\forall i, j \geq 0$. then M is a "graded module".

How do we give a "grade" on the ~~Weyl algebra~~ $A_n(k)$?

The degree doesn't work

because the relation is not homogeneous.

- Filtrations (to solve the problem above!)

R: k-alg., $\mathcal{F}$: filtration on R, is:

$F_0 \subset F_1 \subset \cdots$

All are k-subspaces of R, s.t.

① $R = \bigcup_{i \geq 0} F_i$ and

② $F_i F_j \subset F_{i+j}$, $\forall i, j \geq 0$.

Two filtrations on $A_n(k)$:

① Bernstein Filtration $\mathcal{B} = \{B_i\}_{i \geq 0}$.

by degree $\leq i$.

② Order filtration: $C_i = \left\{ \begin{array}{l} \text{Operators in } A_n(k) \text{ with} \\ \text{order} \leq i \end{array} \right\}$

We are going to exclusively consider B.

Reason: Each B_i is finite dim. k-vector space.

Filtration $\xrightarrow{?}$ grade algebra.

Suppose R is a k-alg., $\mathcal{F}$ a filtration.

$$gr^{\mathcal{F}}(R) = \bigoplus_{i \geq 0} F_i/F_{i-1} \qquad (F_{-1} = 0).$$

Direct sum of k-vector space. How about multiplications?

~~Let $F(i) = F_i/F_{i-1}$~~. Define it Natually!

Not hard too see it is well defined. Extend by k-linear.

What is $gr^{B}(A_n(k))$, B the Bernstein-filtration.

Theorem: $gr^{B}(A_n(k)) \simeq$ poly over k in $2n$-variables.

* $x_i d_i - d_i x_i = 1$,

F_k/F_{k-1} send 1 to 0, so $x_i d_i = d_i x_i$, comm!

It takes some work to show there's no other relations. Argue by contradiction.

* $A_n(k)$ is interesting because $[\bar{x}_i, d_i] \neq 0$.

So the filtration is not that interesting ... maybe I'm wrong ...

* Some "1" is killed, some not.

$x_i d_i - d_i x_i = -1$, is killed. The reason is, when you calculate "$x_i d_i - d_i x_i$", implicitly there is a degree "2" in the formula. Everything with lower degree is killed. This gives a way of killing sth. safe without destroying the structure. In some sense we module the "1" in a different way.

* What if the relation is, say $x^2 d_i - d_i x_i = 1$ and we want to kill the "1"?

or, let R_2 R_3 share the same grade?

Either, find a "grade", so $x^2 d_i$ and $d_i x_i$ have the same grade. Or, find a way to kill "1" without using grades.