

Chap 12

①

1. Tensor Product.

1. Bimodule:

M : Left R -module
Right S - -

M is R - S bimodule if:

$$\forall r \in R, s \in S, m \in M. \quad r(ms) = (rm)s$$

Ex: ① If R is a subring of S , then
 S^k is a S - R -^{bi}module.

② Every ^{left} R -module is an R - \mathbb{Z} -^{bi}module.

③ $A_1 \searrow K[x] \swarrow K[x]$ is NOT bimodule.

Tensor Product:

M : R - S -bimodule, N : S - T -bimodule.

Now we define $M \otimes_S N$

(1) $\mathcal{A} = \left\{ \sum a_i (m_i, n_i) \mid m_i \in M, n_i \in N. \right\}$ Formal finite sums

(2) $r \in R, t \in T,$

(2)

$$r(u, v) = (ru, v), \quad (u, v)t = (u, vt).$$

So A becomes an R - T -bimodule.

(3). B : subgroup of A generated by:

$$(u+u', v) - (u, v) - (u', v)$$

$$(u, v+v') - (u, v) - (u, v')$$

$$(u, sv) - (us, v)$$

By def. of module, bimodule, B is a sub-bimodule of A .

$$M \otimes_S N \triangleq A/B.$$

Universal property of $M \otimes_S N$.

M : R - S -bimodule, N : S - T -bimodule.

L : R - T -bimodule.

$\phi: M \times N \rightarrow L$ is:

① bilinear if $\phi(ru+u', v) = r\phi(u, v) + \phi(u', v)$
 $\phi(u, vt+v') = \phi(u, v)t + \phi(u, v')$

② Balanced if $\phi(us, v) = \phi(u, sv)$

(3)

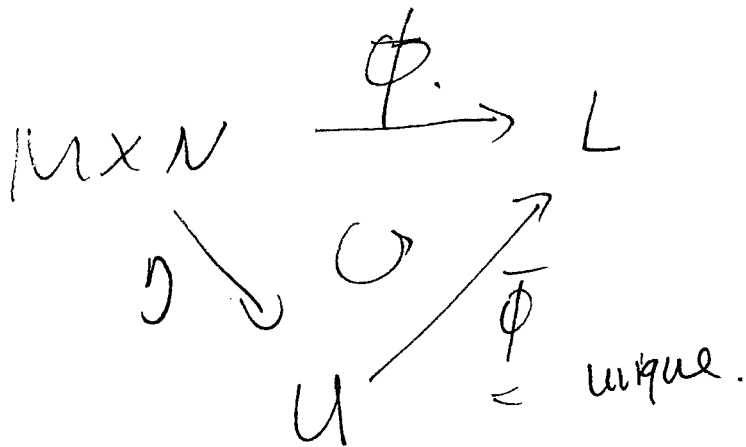
$\eta: M \times N \rightarrow U$: bilinear, balanced.

Say η is "universal" bilinear balanced if:

$\forall \phi: M \times N \rightarrow L$ bilinear, balanced,

$\exists!$ R - T -bimodule homo:

$\bar{\phi}: U \rightarrow L$: s.t.



Thm: ①: $M \times N \rightarrow M \otimes_S N$:

$\pi(u, v) = u \otimes v$ is universal.

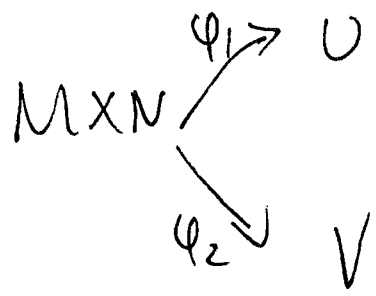
② If ~~$M \otimes_S N$~~ $M \times N \rightarrow U$ is universal.

then $U \cong M \otimes_S N$ as R - T -bimodules.

(skip ①)

~~Growth of Algs and G. k, dim.~~

Proof of ②

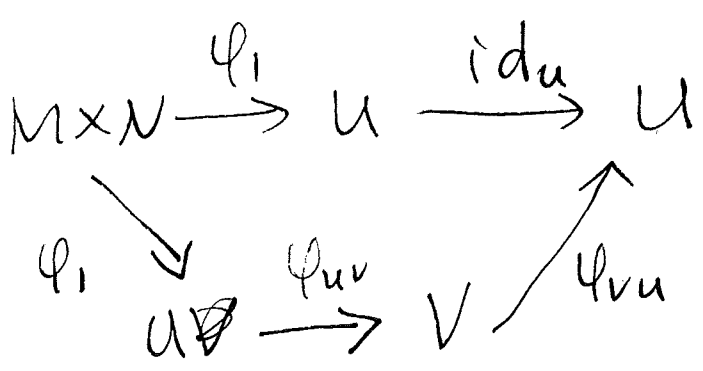
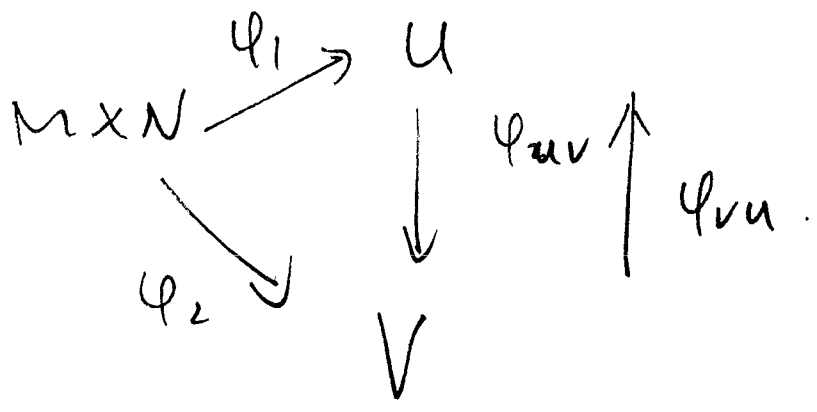


both univ.

$U \cong V$ as R-T-bi

① By univ. prop. of U : $\exists \varphi_{uv}$.

② V : $\exists \varphi_{vu}$.



By univ. prop. of U :
 $\varphi_{uv} \circ \varphi_{vu} \circ \varphi_{uv} = id_U$.

Similarly: $\varphi_{vu} \circ \varphi_{uv} = id_V$.
So ...

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If M, M' are R - S -bi
 N, N' are S - T -bi

$\phi: M \rightarrow M', \psi: N \rightarrow N'$ bimodule homo.

then: $\exists (\phi \otimes \psi)(u \otimes v) = \phi(u) \otimes \psi(v)$ is a
 bimodule homo.

Basic Properties:

1. $M: R$ - S -bi.

$$R \otimes_R M \cong M \cong M \otimes_S S \text{ as bi.}$$

$$2. M \otimes_S \left(\bigoplus_{i \in I} N_i \right) \cong \bigoplus_{i \in I} (M \otimes_S N_i)$$

N_i are S - T -bi

$$3. M_1 \otimes_{R_2} (M_2 \otimes_{R_3} M_3) \cong (M_1 \otimes_{R_2} M_2) \otimes_{R_3} M_3$$

(M_i are R_i - R_{i+1} -bi.)

4. ~~$M \otimes_S N$~~ If R is commutative, M, N are R -modules.

$$\text{then } M \otimes_R N \cong N \otimes_R M.$$

5. Right exactness: if

$$M' \xrightarrow{\psi} M \xrightarrow{\phi} M'' \rightarrow 0 \text{ is an exact sequence of}$$

S-T-bimodule, $B: R-S\text{-bi}$, then

$$B \otimes_S M' \xrightarrow{1 \otimes \psi} B \otimes_S M \xrightarrow{1 \otimes \phi} B \otimes_S M'' \rightarrow 0 \text{ is exact as } R-T\text{-bi}$$

Localization:

$p \in K[x]$. nonzero.

$K[x, p^{-1}] =$ Rational Functions with denominator a power of p .

If $M: K[x]$ -module, then:

$$M[p^{-1}] \cong K[x, p^{-1}] \otimes_{K[x]} M.$$

Ex: $M = K[x] / p^i \cdot K[x]$

then $M[p^{-1}] = 0$.

Prop: $M: K[x]$ -module, $u \in M: 1 \otimes u = 0$ in $M[p^{-1}]$, then $\exists k \geq 0, p^k u = 0$.

pf.

5.5

$$F \xrightarrow{\theta} K[x]^2 \rightarrow M \rightarrow 0.$$

$$K[x, p^{-1}] \otimes_{K[x]} F \rightarrow K[x, p^{-1}] \otimes_{K[x]} K[x]^2 \rightarrow K[x, p^{-1}] \otimes_{K[x]} M \rightarrow 0.$$

||

$$K[x, p^{-1}]^2$$

$$M[p^{-1}] \simeq K[x, p^{-1}]^2 / \mathbb{D} \cdot K[x, p^{-1}] F.$$

as $K[x]$ -modules.

$$\text{Let } \theta(v) = u \quad v \in K[x]^2, \quad u \in F.$$

$$\text{then } (1 \otimes \theta)(1 \otimes v) = 1 \otimes u = 0.$$

$$\text{So, } v \in K[x, p^{-1}] F$$

$$\text{So } \exists p^k \text{ s.t. } p^k v \in F.$$

$$\text{So } p^k u = 0.$$

Question: ~~Suppose \exists good filtration~~ (6)
~~on M , how to (w.r.t.)~~

Suppose M is left A_n -module

M has a good ~~filtration~~ filtration w.r.t. B on A_n .

How to extend it to $M[p^{-1}]$?

Need: $\partial_i(p^{-k} \otimes u) \stackrel{\Delta}{=} p^{-k-1} (p \partial_i u - k \partial_i(p) u)$

Well defined? ~~Suppo~~

$\xrightarrow{\text{①}}$ ① Suppose $m > n$.

$$\partial_i(p^{-m} \otimes p^{m-n} u) \neq \partial_i(p^{-n} \otimes u)$$

② If $p^{-k} \otimes u = 0$ in $M[p^{-1}]$.

$$\partial_i(p^{-k} \otimes u) \neq 0. \quad (\text{By prop})$$

③ Every element in $M[p^{-1}]$ can be written as $p^{-k} \otimes u$,
 some $u \in M$.

$$\Omega_k = \left\{ p^{-k} \otimes u \mid u \in \Gamma_{(m+1)}(K). \right\} \quad (\text{deg } p = m).$$

filtration.

Thm. If M is holonomic, A_n -module,
then so is ${}^M A[P^{-1}]$.

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Pf. $\dim_K \Omega_K \leq \dim_K \Gamma_{(m+1)K}$

So $\dim_K \Omega_K \leq \chi(K^{(m+1)}, M, \Gamma) \leftarrow \text{deg } n$.

So $M[P^{-1}]$ is holonomic.

13. External product: 1. for alg. ①

A, B . k -alg. $A \otimes_k B$ is k -vector space.

if we let $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$,

then $A \otimes_k B$ becomes a k -alg.

denoted by $A \hat{\otimes}_k B$.

Thm 1. R : k -alg. A, B sub alg. of R ,

Suppose: ①. $R = AB$

②. $[A, B] = 0$

③. $\exists k$ bases $\{a_i, i \in \mathbb{N}\}, \{b_j, j \in \mathbb{N}\}$
of A, B , s.t. $\{a_i b_j, i, j \in \mathbb{N}\}$ is
a k -basis of R

then $R \cong A \hat{\otimes}_k B$.

Corollary: $k[x] \hat{\otimes}_k k[y] = k[x, y]$.

$A_m \hat{\otimes}_k A_n = A_{m+n}$.

2. For Modules

~~A, B~~ A, B are K -alg.

~~M, N~~ M, ~~N~~ are left A-modules

N: left B module.

then: $M \otimes_K N$ is $A \hat{\otimes}_K B$ module.

$a \otimes b \in A \hat{\otimes} B$ acts on ~~$m \otimes v$~~ $u \otimes v \in M \otimes_K N$ by

$(a \otimes b) \cdot (u \otimes v) = au \otimes bv.$ } denote by $M \hat{\otimes}_K N$.

Prop 2.1 If M, N are F.G. modules, then so is $M \hat{\otimes} N$.

Lemma 2.2. I: left ideal of A_m
J: - - - A_n .

$A_{m+n} I + A_{m+n} J$ left ideal of A_{m+n} . generated by I & J.

Thm. $(A_m/I) \hat{\otimes} (A_n/J) \cong A_{m+n}/A_{m+n}I + A_{m+n}J$. ③

① $A_{m+n} \rightarrow (A_m/I) \hat{\otimes} A_n/J$ left
as A_{m+n} -~~left~~
module

$$ab \mapsto (a+I) \hat{\otimes} \cancel{b+J} \rightarrow (b+J)$$

\Downarrow

$$A_{m+n}/A_{m+n}I + A_{m+n}J \xrightarrow{\varphi_1} (A_m/I) \hat{\otimes} A_n/J.$$

② $(A_m/I) \hat{\otimes} A_n/J \rightarrow A_{m+n}/A_{m+n}I + A_{m+n}J.$

$$(a+I) \hat{\otimes} \cancel{b+J} \xrightarrow{\varphi_2} ab \cancel{+ I + J}.$$

③ $\varphi_2 \circ \varphi_1 = \text{id} \Rightarrow \varphi$ is ~~proj~~ inj.

+ φ_1 is surj.

$\Rightarrow \varphi_1$ is iso.

Cor: $(A_m/I) \hat{\otimes} A_n \cong A_{m+n}/A_{m+n}I$

as $A_{m+n} - A_n$ -bi module I : left ideal of A_m

Graduations & Filtrations :

① $B_t(A_{m+n}) = \sum_{p+q=t} B_p(A_m) B_q(A_n)$

② $\Gamma_k(M \hat{\otimes} N) = \sum_{i+\hat{j}=k} \Gamma_i(M) \otimes_k \Gamma_{\hat{j}}(N)$

- ① F. D. dim.
- ② $M \hat{\otimes} N = \cup \Gamma_k$
- ③ $B_t \circ \Gamma_k = \Gamma_{k+t}$
 $t \gg 0$

form a good fil. as an A_{m+n} -module

③: $S_n = \text{gr}^B A_n = \bigoplus_{i \geq 0} (\Gamma_i / \Gamma_{i-1})$

$S_{n+m}(t) = \bigoplus_{p+q=t} \cancel{S_p(A_m) S_q(A_n)} S_m(p) S_n(q)$

Since

they are polynomial rings

Write $\text{gr}_{\hat{k}}(M) = \Gamma_k(M) / \Gamma_{k-1}(M)$

$\text{gr}_k(M \hat{\otimes} N) \cong \bigoplus_{i+\hat{j}=k} \text{gr}_i(M) \otimes_k \text{gr}_{\hat{j}}(N)$

as k -vector space.

$\bigoplus_{i+\hat{j}=k} \Gamma_i(M) \otimes \Gamma_{\hat{j}}(N)$
 $\rightarrow \bigoplus \text{gr}_i(M) \otimes \text{gr}_{\hat{j}}(N)$
 with kernel $\Gamma_{i-1} \otimes \Gamma_{\hat{j}} + \Gamma_i \otimes \Gamma_{\hat{j}-1}$
 lift to $\Gamma_k(M \hat{\otimes} N)$
 Γ_{k-1}
 (bijection)

$\text{gr}(M \hat{\otimes} N) \rightarrow \text{gr}(M) \hat{\otimes} \text{gr}(N)$ is iso. as an S_{m+n} module.

Thm 4.1

(5)

M . F.G. A_m -module

N - - A_n - -

Then: $d(M \hat{\otimes} N) = d(M) + d(N)$

$$\binom{d(M)+d(N)}{d(M)}$$

$m(M \hat{\otimes} N) \subseteq m(M) m(N) ??$ $\binom{d(M)}{d(M)+d(N)}$

~~proof: $\dim_k T_k(M \hat{\otimes} N) = \sum_{r=0}^k \sum_{i+j=r} \dim_k \text{gr}_i(M) \dim_k \text{gr}_j(N)$~~

~~So: $\leq \sum_{r=0}^k \dim_k \text{gr}_r(M) \cdot \sum_{j=0}^k \dim_k \text{gr}_j(N)$~~

~~$\dim_{T_k} (M \hat{\otimes} N) \leq \dim$~~

$T_k(M \hat{\otimes} N) \subseteq T_k(M) T_k(N) \subseteq T_{2k}(M \hat{\otimes} N)$

So: $\chi(k, M \hat{\otimes} N) \leq \chi(k, M) \chi(k, N) \leq \chi(2k, M \hat{\otimes} N)$

$\Rightarrow d(M \hat{\otimes} N) = d(M) + d(N)$

$\Rightarrow m(M \hat{\otimes} N) \subseteq m(M) m(N) \binom{d(M)}{d(M)+d(N)}$

Cor. M : holonomic A_m -module

②

N - - - A_n - -

$M \hat{\otimes} N$ - - A_{m+n} -