Chap 12

~~I~~. Tensor Product.

1. Bimodule:

M: Left R-module
right S - -

M is R-S bimodule if:

$r(ms) = (rm)s \quad \forall r \in R, s \in S, m \in M.$

Ex: ① If R is a subring of S, then
S^k is a S-R-bimodule.

② Every left R-module is an R-$\mathbb{Z}$-bimodule.

③ $A_1 \curvearrowright K[x] \curvearrowleft K[x]$ is NOT bimodule.

Tensor Product:

M: R-S-bimodule, N: S-T-bimodule.

Now we define $M \otimes_S N$

(1) $\mathcal{A} = \{ \sum a_i (m_i, n_i) \mid m_i \in M, n_i \in N. \}$. Formal finite sums

(2) $r \in R$, $t \in T$,

$r(u,v) = (ru, v)$, $(u,v)t = (u, vt)$.

So $\mathcal{A}$ becomes an R-T-bimodule.

(3). $\mathcal{B}$: subgroup of $\mathcal{A}$ generated by:

$(u+u', v) - (u,v) - (u', v)$

$(u, v+v') - (u,v) - (u, v')$

$(u, sv) - (us, v)$

By def. of module, bimodule, $\mathcal{B}$ is a sub-bimodule of $\mathcal{A}$.

$$M \underset{S}{\otimes} N \triangleq \mathcal{A}/\mathcal{B}.$$

Universal property of $M \underset{S}{\otimes} N$.

M: R-S-bimodule, N: S-T-bimodule.

L: R-T-bimodule.

$\phi: M \times N \to L$ is:

① bilinear if $\phi(ru+u', v) = r\phi(u,v) + \phi(u', v)$
$\phi(u, vt+v') = \phi(u,v)t + \phi(u, v')$.

② Balanced if $\phi(us, v) = \phi(u, sv)$

$\eta: M \times N \to U$: bilinear, balanced.

Say η is "universal" bilinear balanced if:

$\forall\ \phi: M \times N \to L$ bilinear, balanced,

$\exists!$ R-T-bimodule homo:

$\bar{\phi}: U \to L$: s.t.

$$\begin{tikzcd} M \times N \arrow[r, "\phi"] \arrow[d, "\eta"'] & L \\ U \arrow[ur, "\bar{\phi}"'] \arrow[u, phantom, shift left=4ex, "\circlearrowleft"] & \end{tikzcd}$$

$\bar{\phi}$ unique.

Thm: ① $M \times N \to M \underset{S}{\otimes} N$:

$\pi(u,v) = u \otimes v$ is universal.

② If ~~$M \underset{S}{\otimes} N$~~ $M \times N \to U$ is universal.

then $U \cong M \underset{S}{\otimes} N$ as R-T-bimodules.

(skip ①)

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~~Growth of Algs and G. K. dim.~~

<u>proof of ②</u>.

$$\begin{tikzcd} & U \\ M\times N \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\ & V \end{tikzcd}$$

both univ.

$U \simeq V$ as R-T-bi

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① By univ. prop. of U: $\exists\ \varphi_{uv}$.

② — — — V: $\exists\ \varphi_{vu}$.

$$\begin{tikzcd} & U \arrow[d, shift right, "\varphi_{uv}"'] \\ M\times N \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & V \arrow[u, shift right, "\varphi_{vu}"'] \\ & \end{tikzcd}$$

$$\begin{tikzcd} M\times N \arrow[r, "\varphi_1"] \arrow[d, "\varphi_1"'] & U \arrow[r, "id_u"] & U \\ U \arrow[r, "\varphi_{uv}"] & V \arrow[ur, "\varphi_{vu}"'] & \end{tikzcd}$$

By univ. prop. of U:

$\cancel{\varphi_{uv}\circ}\ \varphi_{vu}\circ\varphi_{uv} = id_u$.

<u>Similarly</u>: $\varphi_{uv}\circ\varphi_{vu} = id_v$.

so . .

If M, M' are R-S-bi
N, N' - S-T-bi

$\phi: M \to M'$, $\psi: N \to N'$ bimodule homo.

then: $\exists\ (\phi \otimes \psi)(u \otimes v) = \phi(u) \otimes \psi(v)$ is a bimodule homo.

Basic Properties:

1. M: R-S-bi.

   $R \otimes_R M \cong M \cong M \otimes_S S$ as bi.

2. $M \otimes_S \left(\bigoplus_{i \in I} N_i \right) \cong \bigoplus_{i \in I} \left(M \otimes_S N_i \right)$

   N_i are S-T-bi

3. $M_1 \otimes_{R_2} \left(M_2 \otimes_{R_3} M_3 \right) \cong \left(M_1 \otimes_{R_2} M_2 \right) \otimes_{R_3} M_3$

   (M_i are R_i-R_{i+1}-bi.

4. ~~$M \otimes_S$~~ If R is commutative, M, N are R-modules.

   then $M \otimes_R N \cong N \otimes_R M$.

5. Right exactness: if

$M' \xrightarrow{\psi} M \xrightarrow{\phi} M'' \longrightarrow 0$ is an exact sequence of

S-T-bimodule, B: R-S-bi, then

$B \otimes_S M' \xrightarrow{1 \otimes \psi} B \otimes_S M \xrightarrow{1 \otimes \phi} B \otimes_S M'' \longrightarrow 0$ is exact as R-T-bi

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Localization:

$p \in k[x]$. nonzero.

$k[x, p^{-1}]$ = Rational Functions with denominator a power of p.

If M: $k[x]$-module, then:

$M[p^{-1}] \triangleq k[x, p^{-1}] \otimes_{k[x]} M$.

Ex: $M = k[x] / p^i \cdot k[x]$

then $M[p^{-1}] = 0$.

Prop: M: $k[x]$-module, $u \in M$: $1 \otimes u = 0$ in $M[p^{-1}]$, then $\exists\, k \geq 0$, $p^k u = 0$.

Pf.

$$F \to k[x]^2 \xrightarrow{\theta} M \to 0.$$

$$k[x,p^{-1}] \otimes_{k[x]} F \to k[x,p^{-1}] \otimes_{k[x]} k[x]^2 \to k[x,p^{-1}] \otimes_{k[x]} M \to 0$$

$$\|$$

$$k[x,p^{-1}]^2$$

$$M[p^{-1}] \simeq k[x,p^{-1}]^2 / k[x,p^{-1}]F.$$

as $k[x]$-modules.

Let $\theta(v) = u$ $v \in k[x]^2$,

then $(1\otimes\theta)(1\otimes v) = 1\otimes u = 0$.

So, $v \in k[x,p^{-1}]F$

So $\exists$ p^k. s.t. $p^k v \in F$.

So $p^k u = 0$.

Question: ~~Suppose $\exists$ good filtration on M, How to (w.r.t.~~

Suppose M is left A_n-module

M has a good ~~filt~~ filtration. w.r.t. $\mathcal{B}$ on A_n.

How to extend it to $M[p^{-1}]$?

Need: $\partial_i(p^{-k}\otimes u) \triangleq p^{-k-1}\otimes(p\,\partial_i u - k\,\partial_i(p)u)$.

Well defined? ~~Suppo~~

$\xrightarrow{f} P$ ①:

① Suppose $m>n$.

$$\partial_i(p^{-m}\otimes p^{m-n}u) \overset{?}{=} \partial_i(p^{-n}\otimes u)$$

② If $p^{-k}\otimes u = 0$ in $M[p^{-1}]$.

$$\partial_i(p^{-k}\otimes u) \overset{?}{=} 0. \quad (\text{By prop})$$

③ Every element in $M[p^{-1}]$ can be written as $p^{-k}\otimes u$, some $u\in M$.

$$\Omega_k = \{\, p^{-k}\otimes u \mid u\in \Gamma_{(m+1)(k)} \,\}. \quad (\deg p = m).$$

filtration.

<u>Thm</u>. If M is holonomic A_n-module, then so is $M[p^{-1}]$. ⑦

<u>Pf</u>: $\dim_K \Omega_K \le \dim_K \Gamma_{(m+1)k}$

So $\dim_K \Omega_K \le \chi(k(m+1), M, \Gamma)$. ← deg n.

So $M[p^{-1}]$ is holonomic.

13. external product: 1. for alg. ①

A, B. k-alg. $A \otimes_k B$ is k-vector space.

if we let $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$.

then $A \otimes_k B$ becomes a k-alg.

denoted by $A \hat{\otimes}_k B$.

Thm 1. R: k-alg. $A \cdot B$ sub alg. of R,

Suppose: ① $R = AB$

② $[A, B] = 0$

③ $\exists$ k bases $\{a_i, i \in N\}$. $\{b_j : j \in N\}$.
of $A \cdot B$. s.t. $\{a_i b_j, i, j \in N\}$ is
a k-basis of R

then $R \cong A \hat{\otimes}_k B$.

Corollary: $k[X] \hat{\otimes}_k k[Y] = k[X, Y]$.

$A_m \hat{\otimes}_k A_n = A_{m+n}$.

2. For Modules

~~M~~: A. B are k-alg.

M. ~~N are~~ left A-modules

N: left B-module.

then: $M \otimes_k N$ is $A \hat{\otimes}_k B$ module.

$a \otimes b \in A \hat{\otimes} B$ acts on ~~$m\otimes$~~ $u \otimes v \in M \otimes_k N$ by

$(a \otimes b) \circ (u \otimes v) = au \otimes bv$. denote by $M \hat{\otimes}_k N$.

Prop 2.1 If M. N are F. G. modules, then so is $M \hat{\otimes} N$.

Lemma 2.2. I: left ideal of A_m

J: — — — A_n.

$A_{m+n} I + A_{m+n} J$ left ideal of A_{m+n}. generated by I & J.

<u>Thm</u>: $(A_m/I)\ \hat{\otimes}\ (A_n/J) \cong A_{m+n}/A_{m+n}I + A_{m+n}J$.

① $A_{m+n} \longrightarrow A_m/I\ \hat{\otimes}\ A_n/J$ as A_{m+n}- ~~lef~~ left Module

$ab \longmapsto (a+I)\ \hat{\otimes}\ (b+J)$

$\Downarrow$

$$A_{m+n}/A_{m+n}I + A_{m+n}J \xrightarrow{\varphi_1} A_m/I\ \hat{\otimes}\ A_n/J.$$

② $A_m/I\ \hat{\otimes}\ A_n/J \longrightarrow A_{m+n}/A_{m+n}I + A_{m+n}J$.

$(a+I)\ \hat{\otimes}\ b \otimes J \xmapsto{\varphi_2} ab + I + J$.

③. $\varphi_2 \circ \varphi_1 = id. \Rightarrow \varphi$ is ~~pro~~ inj.

$+\ \varphi_1$ is surj.

$\Rightarrow\ \varphi_1$ is iso.

<u>Cor</u>: $(A_m/I)\ \hat{\otimes}\ A_n \cong A_{m+n}/A_{m+n}I$

as A_{m+n} - A_n - bimodule I: left ideal of A_m

Graduations & Filtrations :

① $B_t(A_{m+n}) = \sum_{p+q=t} B_p(A_m) B_q(A_n)$

② $\Gamma_k(M \hat{\otimes} N) = \sum_{i+j=k} \Gamma_i(M) \otimes_k \Gamma_j(N)$

form a good fil. as an A_{m+n}-module

(① F. dim. ② $M \hat{\otimes} N = \bigcup \Gamma_k$. ③ $B_t \circ \Gamma_k = \Gamma_{k+t}$ $t \gg 0$.)

③: $S_n = gr^B A_n = \bigoplus_{i \geq 0} (\Gamma_i / \Gamma_{i-1})$

$S_{n+m}(t) = \bigoplus_{p+q=t} \cancel{S_p(A_m) S_q(A_n)}$. $S_m(p) S_n(q)$

they are polynomial ~~rig~~ rings

Write $gr_k(M) = \Gamma_k(M) / \Gamma_{k-1}(M)$

$gr_k(M \hat{\otimes} N) \triangleq \bigoplus_{i+j=k} gr_i(M) \otimes_k gr_j(N)$

as K-Vector space.

Since $\bigoplus_{i+j=k} \Gamma_i(M) \otimes \Gamma_j(N) \to \bigoplus gr_i(M) \otimes gr_j(N)$. ~~Lift~~ with kernel $\Gamma_{i-1} \otimes \Gamma_j + \Gamma_i \otimes \Gamma_{j-1}$ lift to $\Gamma_k(M \hat{\otimes} N) / \Gamma_{k-1}$. (bijection)

$gr(M \hat{\otimes} N) \to gr(M) \hat{\otimes} gr(N)$ is iso. as an S_{m+n} module.

<u>Thm 4.1</u>

M. F.G. A_m-module

N — — A_n — —

$$\binom{d(M)+d(N)}{d(M)}.$$

Then: $d(M\hat{\otimes} N) = d(M) + d(N)$

$$m(M\hat{\otimes} N) \leq m(M)\, m(N) \, ??\, C^{d(M)}_{d(M)+d(N)}$$

<u>proof</u>: $\cancel{\dim_k T_k(M\hat{\otimes} N) = \sum_{r=0}^{k} \sum_{i+j=r} \dim_k gr_i(M) \dim_k gr_j(N).}$

~~So~~: $\cancel{\leq \sum_{i=0}^{k} \dim_k gr_i(M) \cdot \sum_{j=0}^{k} \dim_k gr_j(N)}$

$\cancel{\dim_{T_k}(M\hat{\otimes} N) \leq \dim}$

$$T_k(M\hat{\otimes} N) \subseteq T_k(M)\, T_k(N) \subseteq T_{2k}(M\hat{\otimes} N)$$

So:

$$\chi(k, M\hat{\otimes} N) \leq \chi(k, M)\, \chi(k, N) \leq \chi(2k, M\hat{\otimes} N)$$

$$\Rightarrow d(M\hat{\otimes} N) = d(M) + d(N)$$

$$\Rightarrow m(M\hat{\otimes} N) \leq m(M)\, m(N)\, C^{d(M)}_{d(M)+d(N)}.$$

<u>Cor.</u> M: holonomic A_m-module

N - - - A_n - -

$M \hat{\otimes} N$ - - A_{m+n} -