2012. 9. 17.

Messing

X: smooth, alg. variety. $/K$, char 0.

$\mathcal{D}_X$ sheaf of rings of diff. operators

$$\mathcal{D}_X = \bigcup_{m \geq 0} \mathcal{D}_{X,m}$$ ← at most m

$$\mathcal{D}_{X,m} \cdot \mathcal{D}_{X,n} \subset \mathcal{D}_{X,m+n}.$$

$$[\mathcal{D}_{X,m}, \mathcal{D}_{X,n}] \subset \mathcal{D}_{X,m+n-1}.$$

$$gr(\mathcal{D}_X) = \bigoplus_{m \geq 0} \mathcal{D}_{X,m} / \mathcal{D}_{X,m-1}.$$

$$\| $$

$$sym(\Theta_X)$$

↑ tangent bundle.

$spec(gr(\mathcal{D}_X)) = T_X^*$.

contangent bundle.

M: coherent $\mathcal{D}_X$-module,

locally on X, $\exists$ $\underbrace{\text{filtration}}_{\text{increasing}}$ $(F_m)_{m\geq 0}$ of M,

s.t. $\mathcal{D}_{X,m} F_n \subset F_{m+n}$.

Said to be "good filtration": of $gr_{F.}(M)$ is

a coherent $gr(\mathcal{D}_X)$-module.

Annihilator of $gr_{F.}(M) \subset gr(\mathcal{D}_X)$.

Let $J_M = \sqrt{\mathrm{Ann}}$.

"J_M is stable under the bracket":

$P, Q \in \mathcal{D}_X$. $P \in \mathcal{D}_{X,m}$, $Q \in \mathcal{D}_{X,n}$.

$\sigma(P), \sigma(Q)$: $\sigma(P) \in \mathcal{D}_{X,m}/\mathcal{D}_{X,m-1}$, $\sigma(Q) \in \mathcal{D}_{X,n}/\mathcal{D}_{X,n-1}$.

$[\sigma(P), \sigma(Q)] = \sigma(P\circ Q - Q \circ P) \in \mathcal{D}_{m+n-1}/\mathcal{D}_{m+n-2}$.

T^*_X: symplectic Manifold.

$x_1 \sim x_n$, local coordinates

$\frac{\partial}{\partial x_i} \mapsto = \theta_i$.

$\alpha = \sum \theta_i \frac{\partial}{\partial x_i}$

$d\alpha = \sum d\theta_i \wedge dx_i$

$p \in T^*(X)$, $T_p(T^*_X)$ symplectic form

$V = \mathrm{spec}(gr(\mathcal{D}_X)/J_M)$: closed subschem of T^*_X. then

$T_p V \subset T_p(T^*_X)$.

Then conditions about the [ ] garentees that

if p is smooth in V, then

$T_p(V)^\perp \subset T_p(V)$. ~~(??)~~

$\Rightarrow \dim T_p(V) \geq n = \dim X$. (because $\dim T^*_X = 2n$)

M is ~~holomophic~~ holonomic D-module, if for all p.

$\dim T_p(V) = n$.

Corresponding to. "Maximally over ~~determined~~ determined.." (??)
A system of PDEs.

Especially, O-module is holonomic.

"Coherent + dim ~~restrictio~~ condition

Regularity. Assume M is holonomic. D-module on X,

Fact: $\exists$ dense open set U in X, s.t.

$M|_U$ is coherent $\mathcal{O}_U$-module

$\Rightarrow$ locally free, so a vector bundle on U.

Now let $i : C \to X$ ~~has~~ be a morphism wher

C is a smooth ~~Automorphism~~ curve $/k$, $C \cap U \neq \phi$,

$i^*(M/U)$ vector bundle of $i^{-1}(U) \cap \ldots$ with ...

M is regular provided $i^*(M/U)$ is.. (ind. of choice of C)

Derived Categories.

Let $\mathfrak{A}$: abelian Category.

Consider complexes ~~$K^\cdot$~~ $K^\cdot$: ~~orb~~ objects of $\mathfrak{A}$

$$K^\cdot \overset{u}{\underset{v}{\rightrightarrows}} L^\cdot$$

homotopic if. ..

$K(\mathfrak{a})$: ~~orbjec~~ object of a complexes

$Hom(K^\cdot, L^\cdot)$ = alg. group of homotopy

" " of morphisms

If $s: K^\cdot \to L^\cdot$, then

$H^i_{(s)}: H^i(K^\cdot) \to H^i(L^\cdot)$ is ~~Iso. if H's for all i,~~

Iso ~~if~~ for all i.

then say s is a "quasi-iso"

$D(\mathfrak{a}) = S^{-1}(K(\mathfrak{a}))$

Functor:

$$\begin{tikzcd} K(a) \arrow[r] \arrow[d] & C \\ S^{-1}(K(a)) \arrow[ur, dashed] & \end{tikzcd}$$

factors through. any quasi-iso on C.

$D^b(a) \subset D(a)$. given by $K^\cdot \in K(a)$ s.t.

$H^i(K^\cdot) = 0$ for $i \notin [m,n]$.

$\mathcal{A}$: Quasi-coherent $\mathcal{D}_X$-modules. (as $\mathcal{O}_X$-module)

$D^b(a) \supset D^b_{r.h}(\mathcal{A})$

$K^\cdot$: all $H^i(K^\cdot)$ are regular, holomonic.

Now assume $k = \mathbb{C}$,

X, alg. variety, X^{an}: analytic. Complex manifold.

$\mathcal{D}_{X^{an}}$ similar as $\mathcal{D}_X$, just different in functions and topo.

"Regular Holomonic" extends.

$$M \mapsto M^{an}$$

$$X^{an} \overset{u}{\mapsto} X$$

$$u^*(M) = M^{an}.$$

the functor keeps regular,
Regular holonomic.

$$DR(M^{an}) = \underbrace{R\,Hom_{D_{X^{an}}}(\mathcal{O}_{X^{an}}, M^{an})}.$$

$$\Omega^{\cdot}_{X^{an}} \otimes_{\mathcal{O}_X} (M^{an}) \quad \text{by } \nabla^{an}.$$

$$Sol(M) = R.\,Hom_{D_{X^{an}}}(M^{an}, \mathcal{O}_{X^{an}}).$$

Assume $M^{\cdot}$ complex of $\mathcal{D}_X$-modules

bounded, regular, holonomic.

$DR(M^{\cdot\,an})$ = Complex of sheaves of $\mathbb{C}_{X^{an}}$ vector spaces

↑ Constant sheaf.

$H^i(DR(M^{an})) = 0$ for $i \notin [m,n]$.

For each $i \mid m \le i \le n$, $H^i(DR(M^{\cdot an}))$ is algebraic constructible. i.e.

For each i, $\exists\ Y_0 = \phi \subset Y_1 \subset \cdots \subset Y_k = X$,

Zariski closed subsets,

($Y_j - Y_{j-1}$ locally closed in X) ~~the a~~

$H^i(DR(M^{\cdot an}))\big|_{(Y_j - Y_{j-1})^{an}}$ is locally constant

with finite dim. fibres.

$D^b_{r.h}(X) \xrightarrow{DR} D^b(X^{an})$

$\mathbb{C}_{X^{an}}$, constructible.

Riemann Hilbert corre. states that DR is an equivalence of ~~category~~. categories.

Consider M: $\mathcal{D}_X$-module. 3-9

which is regular, holo.

think of it as a complex by putting it to the 0-dim.

$DR(M^{an})$ is an $D^b(X^{an})$ is a special case

$\mathbb{C}_{X^{an}}$, const.

regular holonomic $\mathcal{D}_X$-module, an abelian category.

$DR_\circ$ (reg. holo. $\mathcal{D}_X$-module).

by def. the ~~Perserse~~ Perverse sheaves.

Question ~~answer~~ by Ionut

$X \xrightarrow{f} Y$. Smooth alg. varieties.

Six Operations:

f^*, f_*, $f^!$, $f_!$, $\otimes^L$, $R\mathcal{H}om$

they exist for D-modules

Stablizes (?) "Regular, holo." ~~sheaves~~.

$\Rightarrow$

$X^{an} \to Y^{an}$

$\cdots$

$\cdots$

$\cdots$