X: smooth alg. variety $/K$ $\quad$ $\mathrm{char}(K)=0$.

$\mathcal{D}_X$: sheaf of rings of diff. ops $\quad$ $\mathcal{D}_X = \bigcup_{m\geq 0} \mathcal{D}_{X,m}$: operators of order at most m

$\mathcal{D}_{X,m}\mathcal{D}_{X,n} \subseteq \mathcal{D}_{X,m+n}$

$[\mathcal{D}_{X,m}, \mathcal{D}_{X,n}] \subseteq \mathcal{D}_{X,m+n-1}$ $\qquad$ $\mathrm{gr.}(\mathcal{D}_X) = \bigoplus_{m\geq 0} \mathcal{D}_{X_m}/\mathcal{D}_{X_{m-1}}$

$\qquad \mathrm{gr.}(\mathcal{D}_X) = \mathrm{Sym}(\Theta_X)$

$\mathrm{Spec}\,(\mathrm{gr.}(\mathcal{D}_X)) = T^*_X$: cotangent bundle.

M: coherent $\mathcal{D}_X$-module

Locally on X, $\exists$ filtration $(F_m)_{m\geq 0}$ of M s.t. (inc.)

$\mathcal{D}_{X,m} F_n \subset F_{m+n}$.

Said to be a good filtration if $\mathrm{gr}_{F.}(M)$ is a coherent $\mathrm{gr.}(\mathcal{D}_X)$-module.

(always exist locally. globally if X quasi-proj.)

Assuming good filtration, Annihilator of $\mathrm{gr}_{F.}(M) \subset \mathrm{gr}(\mathcal{D}_X)$

Let J_M = radical of annih.

J_M is stable under bracket. $P, Q \in \mathcal{D}_X$ say $P \in \mathcal{D}_{X,m}$, $Q \in \mathcal{D}_{X,n}$

then $\sigma(P)$, $\sigma(Q)$ associated symbols

$\sigma(P) \in \mathcal{D}_{X,m}/\mathcal{D}_{X,m-1}$, $\sigma(Q) \in \mathcal{D}_{X,n}/\mathcal{D}_{X,n-1}$ $\qquad$ $[\sigma(P), \sigma(Q)] \overset{\mathrm{def}}{=} \sigma(P\circ Q - Q\circ P)$

$\in \mathcal{D}_{m+n-1}/\mathcal{D}_{m+n-2}$

T_X^* is symplectic manifold. $x_1, \dots, x_n$ loc. on X $\quad \theta_i = \partial/\partial x_i$

$$\alpha_\xi = \sum_{i=1}^{n} \theta_i \, dx_i \qquad d\alpha = \sum d\theta_i \wedge dx_i$$

$p \in T^*(X)$, $T_p(T_X^*)$: tangent space @ p.

Let $V = \operatorname{Spec}(\operatorname{gr} D_X / J_M)$. Then $T_p(V) \subset T_p(T_X^*)$

If p is a smooth pt. of V, $T_p(V)^\perp \subset T_p(V)$

$\Rightarrow \dim T_p(V) \geq n = \dim(X)$ $\quad$ ($\frac{1}{2}$-dimen of $T_p(T_X^*)$)

M is a <u>holonomic</u> $\mathcal{D}$-module if, for all smooth p, $\dim T_p(V) = n$
(assumes M is coherent)

($\Longleftrightarrow$ "maximally overdetermined" system of PDEs. cf. Spencer, Bulletin, mid '60's)

Holonomic $\mathcal{D}$-modules form abelian category.

<u>Fact</u>: $\exists$ dense open set $\mathcal{U}$ in X s.t. ~~$M|_\mathcal{U}$~~ $M|_\mathcal{U}$ is a coherent $\mathcal{O}_\mathcal{U}$-mod
$\Rightarrow$ locally free vector bundle on $\mathcal{U}$. (since $M|_\mathcal{U}$ has infg. conn.)

Let $i: C \to X$ be a morphism $\quad C$: smooth curve $/K$ $\quad C \cap \mathcal{U} \neq \phi$
$i^*(M|_\mathcal{U})$ vector bundle on $i^{-1}(\mathcal{U})$ with infg. conn.

Say M is regular if $i^*(M|_\mathcal{U})$ is regular (just need to check for one non-triv. morphism)

Derived Categories: $\mathcal{A}$: abelian category

consider complexes $K^\bullet$ of objects of $\mathcal{A}$.

Consider $K^\cdot \overset{u}{\underset{v}{\rightrightarrows}} L^\cdot$ where u homotopic to v ($\sim$ relation)

$K(\mathcal{A})$: objects are complexes

$\mathrm{Hom}_{K(\mathcal{A})}(K^\cdot, L^\cdot)$ = ab. gp. of homotopy classes of morphisms.

If $s: K^\cdot \to L^\cdot$ and $H^i(s): H^i(K^\cdot) \to H^i(L^\cdot)$ is isom. for all $i \in \mathbb{Z}$

we say s is a "quasi-isomorphism" from $K^\cdot \to L^\cdot$

derived cat. $D(\mathcal{A}) = S^{-1}(K(\mathcal{A}))$: formally invert quasi-isom.

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r] \arrow[dr] & C \\ & S^{-1}(K(\mathcal{A})) \arrow[u, dashed] \end{tikzcd}$$

C : category in which quasi-isom become isom., then factors through derived cat.

$D^b(\mathcal{A}) \subset D(\mathcal{A})$: complexes w/ bounded cohom.

Given by $K^\cdot \in K(\mathcal{A})$ s.t. $H^i(K^\cdot) = 0$ if $i \notin [m,n]$ finite interval

$\mathcal{A}$ = quasi-coh. $\mathcal{D}_X$-modules

$D^b(\mathcal{A}) \supset$ full subcat $D^b_{\text{reg,hol.}}(\mathcal{A})$: complexes of $\mathcal{D}_X$-modules $K^\cdot$ s.t. all $H^i(K^\cdot)$ are regular, holonomic.

Assume field is $\mathbb{C}$. $X/\mathbb{C}$ has analytic structure as cx. manifold X^{an}. $\mathcal{D}_{X^{an}}$ with coeffs. as holom. functions in classical top.

not alg. func. in Zariski top

Definitions of regular / holonomic same, though proofs in analytic context are more difficult.

Functor from $\mathcal{D}_X$-modules $\longrightarrow$ $\mathcal{D}_{X^{an}}$ modules

$$M \longmapsto M^{an} \qquad \text{for } X^{an} \xrightarrow{\mu} X$$

via $\mu^*(M) = M^{an}$

preserves regularity, holonomicity

$$\mathrm{DRham}(M^{an}) = R\mathcal{H}om_{\mathcal{D}_{X^{an}}}(\mathcal{O}_{X^{an}}, M^{an})$$

from complex $\Omega^{\cdot}_{X^{an}} \otimes_{\mathcal{O}_X} M^{an}$

$$\mathrm{Sol}(M^{an}) = R\mathcal{H}om_{\mathcal{D}_{X^{an}}}(M^{an}, \mathcal{O}_{X^{an}})$$

"solutions to linear systems of PDEs"

Assume $M^{\cdot}$ is a complex of $\mathcal{D}_X$-modules. Bounded, reg., holon.

$\mathrm{DR}(M^{\cdot\, an}) := \mathrm{DRham}(M^{\cdot\, an})$ = cx. of sheaves of $\underbrace{\mathbb{C}_{X^{an}}}_{\text{const shf.}}$-vector spaces

$H^i(\mathrm{DR}(M^{an})) = 0$ for $i \notin [m,n]$. For $m \leq i \leq n$, $H^i(\mathrm{DR}(M^{\cdot\, an}))$ is alg. constructible

This means, for each i, $\exists$ $Y_0 = \phi \subset Y_1 \subset \cdots \subset Y_k = X$ $\quad$ Y_j: Zariski closed

$Y_j - Y_{j-1}$ locally closed in X.

then $H^i(\mathrm{DR}(M^{\cdot\, an}))\big|(Y_j - Y_{j-1})^{an}$ is locally const. with finite-dimensional fibers.

$$D^b_{\text{rg.hol}}(X) \xrightarrow{\text{DR}} D^b_{\mathbb{C}_{X^{an}},\text{const.}}(X^{an})$$

as in $\mathcal{D}_X$

$$M \longmapsto \text{DR}(M^{an})$$

R-H correspondence: This is an equivalence of categories!

Consider M a $\mathcal{D}_X$-module, rg., holonomic (as complex supported in degree 0)

$\text{DR}(M^{an})$ is in $D^b_{\mathbb{C}_{X^{an}},\text{const.}}(X^{an})$

Reg., holon. $\mathcal{D}_X$-modules, apply DR functor, get abelian category in $D^b_{\mathbb{C}_{X^{an}},\text{cons}}(X^{an})$ by transport of structure.

These are, by definition, "perverse sheaves"

Given morphism $f : X \to Y$ smooth alg. vars.

Six Operations: $f^*, f_*, f^!, f_!, \overset{L}{\otimes}, R\mathcal{H}om$

maps on $D^b_{r,h}(X) \to D^b_{r,h}(Y)$

bifunctor

(upper $*$ and $!$ go other way)

same formalism exists on analytic side. DeRham functor is compatible w.r.t. all 6. "Poincaré-Verdier Duality"