9月12日, 2012年 Talk 2

Messing.

Riemann Hilbert Proplem:

$U/\mathbb{C}$, smooth. alg. variety.

M: $\mathcal{O}_U$-module.

a connection on M is the giving of

equivantly $\nabla: M \to \Omega^1_U \otimes_{\mathcal{O}_U} M$.

① $f \in \mathcal{O}_U$, $m \in M$,

① $\nabla(fm) = df \otimes m + f\,\nabla(m)$.

② additive, $\mathbb{C}$-linear.

$\Longleftrightarrow$ Θ_U = tangent bundle = $\mathrm{Hom}(\Omega^1_U, \mathcal{O}_U)$

$\theta \in \Theta_U$, $\nabla_\theta : M \to M$. s.t.

$$\nabla_\theta(fm) = \theta(f)\cdot m + f\cdot \nabla_\theta(m).$$

$$\nabla_\theta = \nabla \circ (\theta \otimes 1_M)$$

$p \geq 1$, $\nabla^p : \Omega_U^p \otimes M \to \Omega_U^{p+1} \otimes M$

$\eta \in \Omega_U^p$, $\nabla^p(\eta \otimes m) = d\eta \otimes m + \eta \wedge \nabla(m)$

Note that: $\wedge \otimes 1_M : \Omega^p \otimes \Omega^1 \otimes M \to \Omega^{p+1} \otimes M$.

Fact: $\nabla^{p+1} \circ \nabla^p = 0$ for all p iff $\nabla^1 \circ \nabla = 0$
$(\nabla^0 \triangleq \nabla)$

Which is equivalent to saying that:

$\forall \theta_1, \theta_2 \in \Theta_U$,

$\nabla_{[\theta_1, \theta_2]} = [\nabla_{\theta_1}, \nabla_{\theta_2}]$.

$\Updownarrow$

integrable connection. $\overset{\text{means}}{\leadsto}$ Curvature $= 0$

So we have the complex of $\mathcal{O}_U$ modules

$\Omega_U^\bullet \otimes M$ under this assumption.

$$H_{DR}(\mathcal{U},(M,\nabla)) = \mathbb{H}^*(\mathcal{U}, \Omega^\bullet_{\mathcal{U}} \otimes M)$$

If $\mathcal{U}$ is affine, then

$$H^j_{DR}(\mathcal{U},(M,\nabla)) = \text{Kernel of } \nabla^j \Big/ \text{Image of } \nabla^{j-1}.$$

(on global sections $\Gamma(\mathcal{U}, \Omega^\bullet_{\mathcal{U}} \otimes M)$.

In particular,

$M = \mathcal{O}_{\mathcal{U}}$, $\nabla(f) = df$. "trivial connection".

$H_{DR}^\bullet(\mathcal{U},(M,\nabla))$ is just the usual de Rham coho.

Grothendieck's thm: 1966 P.M. IHES. Vol 29.

$$H^*_{DR}(U/\mathbb{C}) \cong H^*(U^{an.}, \mathbb{C})$$

Elementary when U is projective. ~~or~~ and more generally proper

Not so when U is affine.

"Resolution of singularities"

(M, ∇), ∇ integrable.

has regular singularities

<u>means</u>: (defined due to Deligne).

$\exists\ X \supset U$, U open in X,

X: proper, smooth, $/\mathbb{C}$ ~~open~~.

$D = X \setminus U$ is a divisor with normal crosses.

meaning $\forall\, x \in D$ processes a Zariski open neighbourhood V, $\exists$ etale morphism

$\pi: V \to \mathbb{A}^n_{\mathbb{C}}$. s.t. $t_1 = \pi^*(T_1), \dots, t_n(\pi^*(T_n))$.

$*$ T_i: coordinate function in $\mathbb{A}^n$

$D \cap V = V(t_1, \dots, t_r)$ for some $1 \le r \le n$.

Assume M coherent, ~~then~~ then $\exists\ \overline{M}$ on X

locally free, $\mathcal{O}_X$-module, s.t.

1) $\overline{M}|_U \simeq M$

2) $\exists$ $\overline{\nabla}: \overline{M} \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \overline{M}$.

a connection. extending ∇.

$$\Omega^1_X(\log D) : \begin{cases} \text{on } U = \Omega^1_X. \\ \text{at point of } D \text{, it has } \sout{\text{basis}} \text{ bases.} \\ \frac{dt_1}{t_1}, \dots, \frac{dt_r}{t_r}, dt_{r+1}, \dots, dt_n \end{cases}$$

(control the poles at $x \in D$).

On U^{an}, $\boxed{an = \text{analytic}}$.

$\mathcal{M}$: $\mathcal{O}_{U^{an}}$ - module.

with "integrable connection"

$$\mathcal{H}^0(\Omega^{\bullet}_{U^{an}} \otimes_{\mathcal{O}_U^{an}} \mathcal{M}) = \ker(\nabla)$$

Ker(∇): local ~~coff~~ coeff system
~~of~~ complex vector space ~~of~~ in U^{an}.

same as rep. of $\pi_1(U^{an}) \xrightarrow{\rho} GL(n, \mathbb{C})$.

Category of (M, ∇), ∇ is integrable, regular.

$M_1 \xrightarrow{\phi} M_2$, $\mathcal{O}_U$-linear,
call ϕ "regular" if $\nabla_{M_2,\theta} \circ \phi = \phi \circ \nabla_{M_1,\theta}$. $\forall \theta \in \Theta_U$.
compatible with connection.

∇^{an}
M^{an}.

<u>Thm</u> (Deligne).

$$(M, \nabla) \longmapsto (M^{an}, \nabla^{an}) \longmapsto \cancel{(M^{an}, U^{an})}$$
rep. of $\pi_1(U^{an})$.

is an equi. of categories.

$$H_{DR}(U,(M,\nabla)) \xrightarrow{\sim} H_{DR}(U^{an}, M^{an}, \nabla^{an})$$ (2-8

$$\wr\|$$

$$H^*(U^{an}, \underset{\substack{\uparrow \\ \text{local system.}}}{M^{an\,\nabla^{an}}}).$$

LNM163

X: Smooth variety $/ K$, ~~a fied~~ a field of char 0.

(just assume $K=\mathbb{C}$ if you wish)

$\mathcal{O}_X$, $\mathcal{E}nd_K(\mathcal{O}_X) \overset{\text{subsheaf}}{\supset} \mathcal{D}_X$.

~~finite~~ diff. operators.

For $n \geq 0$, $\mathcal{D}_n \subset \mathcal{D} = \mathcal{D}_X$ is defined as :

~~For~~ for $P \in \mathcal{E}nd(\mathcal{O}_X)$, $P \in \mathcal{D}_n$, iff.

$\forall f_0, \cdots, f_n \in \mathcal{O}_X$,

$[\cdots[[[P, f_0], f_1] f_2] \cdots f_n] = 0$ | f_i means multiplied by f_i in $\mathcal{E}nd(\mathcal{O}_X)$

X smooth $\Rightarrow \mathcal{D}_X$ is generated by

$$\mathcal{O}_X,\ \Theta_X.$$

$P \in \mathcal{D}_p$, $Q \in \mathcal{D}_q$, then $[P, Q] \in \mathcal{D}_{p+q-1}$

$P \circ Q \in \mathcal{D}_{p+q}$. If we define.

$$gr(\mathcal{D}) \triangleq \bigoplus_{p \geq 0} \mathcal{D}_p / \mathcal{D}_{p-1},$$ then

$gr(\mathcal{D})$ is a commutative ring.

Each $\mathcal{D}_p$ is a coherent $\mathcal{O}_X$-module,

$\mathcal{D}$ is quasi coherent.

$$gr(\mathcal{D}) = Sym_{\mathcal{O}_X}(\Theta_X).$$

$spec(gr(\mathcal{D})) = T^*_X$. contangent bundle of X.

$\underline{\mathfrak{M}}$: Assume it to be a left D_X-module

Assume

$(P_{ij}) \in M_{m,n}(\mathcal{D})$

$$\sum_{j=1}^{n} P_{ij} f_j = 0 \quad \text{for } i = 1, \dots, m.$$

m PDE's.

Define: $\mathcal{D}^m \xrightarrow{P} \mathcal{D}^n$ by $P = (P_{ij})$.

$\vec{R} = (R_1, \dots, R_m) \longmapsto \vec{R} P$.

$$\mathcal{D}^m \xrightarrow{P} \mathcal{D}^n \xrightarrow{\pi} M \longrightarrow 0.$$
$$M \xrightarrow{F} \mathcal{O}.$$

$(F \circ \pi(e_j) = f_j)$

Consider:

F is the solution of

~~$\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{O})$~~

$\mathrm{Sol}(M) = \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{O}_X)$

Consider:

$\mathrm{Ext}^q_{\mathcal{D}}(M, \mathcal{O})$

M a left $\mathcal{D}$-module, having an integrable connection.

i.e. (2-1)

$\theta \in \Theta$, $\nabla_\theta(m) = \theta \cdot m \in M$.

$DR(M) = \Omega_X^\bullet \otimes M$

$\mathcal{D} \otimes_{\mathcal{O}_X} \Lambda^\bullet \Theta$

Resolution of $\mathcal{O}_X$:

$$\mathcal{D} \otimes \Lambda^n \Theta \xrightarrow{\partial} \cdots \longrightarrow \mathcal{D} \otimes \Lambda^n \Theta \xrightarrow{\partial} \underset{\mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}_X}{\mathcal{D}} \xrightarrow{\varepsilon} \mathcal{O}_X .$$

$$\partial\big(P \otimes (\theta_1 \wedge \cdots \wedge \theta_n)\big) = P\theta_1 \otimes (\theta_2 \wedge \cdots \wedge \theta_n) - \sum_{i=1}^{n} P(\theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_n)$$

$\varepsilon(P \otimes f) = P(f)$.

??

$$\mathrm{Ext}^q_{\mathcal{D}}(\mathcal{O}_X, M) = H^q(\Omega_X^\bullet \otimes M)$$