D-Modules

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① A little history. (1960s ~ 1980s) 1-1

Gelfand, ICM 1954. <u>Question</u>:

$P(x_1, \cdots, x_n) \in \mathbb{R}[x_1 \sim x_n]$.

<u>Assume</u>: $P(x) \geq 0$ for all $x \in \mathbb{R}^n$.

$\phi \in C_0^\infty(\mathbb{R}^n)$.

$T_\phi(\lambda)$, $(\lambda \in \mathbb{C},\ Re(\lambda) > 0)$

$\triangleq \int P^\lambda \phi \, dx$, holomorphic when $Re(\lambda) > 0$.

<u>Q</u>: Does it admit analytic continuation ?

to a meromorphic function whose order of poles are bounded independent of ϕ. and whose ~~pl~~ poles in finite union of Arithmetic properties.

1968 Positive! ~~@~~ Atiyah, independently Bernstein · ~~Gelfand~~ · Gelfand

Both proofs use resolution of singularities.

1972. Bernstein gives a second elementary proof.

Explicitly used D-modules

$$A_n = \mathbb{C}[x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}]. \quad \boxed{x_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} x_i = -1}$$

See Paul Garrett's webpage for more details.

1978 - 1980, Kazhdan - Lusztig.

Kazhdan - Lusztig Polynomials.

q: power of prime number.

$\mathbb{Z}[\sqrt{q}, \frac{1}{\sqrt{q}}](T)$.

(W, S): $x, w \in W$, $x \le w$ defined by words expression, "Bruhat order."

S: simple reflections.

$\exists$ Basis. $(e_w)_{w \in W}$. Hecke algebra.

free over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.

K-L polynomial is just the ~~ort~~ other basis.

Verma Modules. M_w

Module over $\mathfrak{g}$. some simply connected semisimple alg. group & Lie algebra.

Unique. alg. quotient L_w.

$\exists$ χ: character,

$$\chi(L_w) = \sum_{x \le w} (-1)^{\ell(w)-\ell(x)} P_{x,w}(1) \cdot \chi(M_x).$$

Conjectured by K-L

(1981) Beilinson, - Bernstein.

ind Brylinski - Kashiwara.

Both solutions make use of Riemann-Hilbert Correspondence. for holonomic regular D-modules.

Alhfors's text book

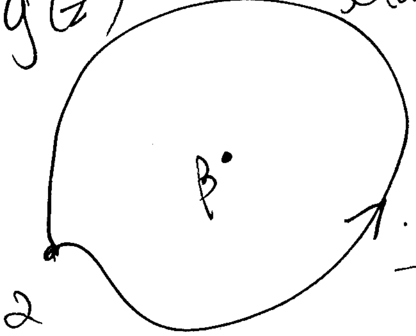
last 12-13 pages.

He explains if we have an equation, say,

$$f''(z) + a_1(z)f'(z) + a_2(z)f(z) = 0$$

a_1, a_2 meromorphic

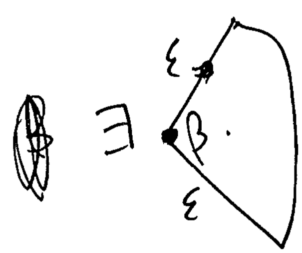
$g(z)$



solution may not come back if $\exists$ pole of a_i in the loop

$\rightarrow$ multi-valued.

But :



radius ε. not the whole circle.

$\star$

$\exists N, C.$

$$|z-\beta|^N |g(z)| \le C.$$

If poles of a_1, a_2 are $0, 1, \infty$.

$g(z)$, the solution satisfies $\star$ iff

$a_1(z)$ has at most a first order pole.

$a_2(z)$ has at most a second order pole.

$$y^2 = x(x-1)(x-\lambda), \quad \boxed{①} \quad \lambda \neq 0, 1.$$

$\Downarrow$

E_λ:

$$zy^2 = x(x-z)(x-\lambda z).$$

$E_\lambda \subset \mathbb{P}^2_{\mathbb{C}}$. is a torus.

$\mathbb{Z}[\frac{1}{2}][\lambda] \subset \mathbb{Q}(\lambda)$.

$K = \mathbb{Q}(\lambda)(x, y)$. x, y satisfies $\boxed{①}$.

$\frac{\partial}{\partial \lambda} \longrightarrow$ extend to K : called D_x. s.t.

(i) $D_x(x) = 0$.

For $\eta = f\,dg$, where $f, g \in K$,

let $D_x(\eta) = D_x(f)\,dg + f\,d(D_x(g))$.

Then: ① D_x is additive

② Product Rule ✓.

③ $D_x(dg) = d(D_x g)$

If η has all residues $= 0$,

Then $\mathrm{Res}_p(D_X(\lambda)) = D_X(\mathrm{Res}_p(\eta)) = 0$ 1-6

So D_X preserves differentials of 2^{nd} kind.

So D_X acts on: $\dfrac{\text{Diff of } 2^{nd} \text{ kind}}{\text{exact}} = H^1_{DR}(E_\lambda)$

$\mathbb{Q}(\lambda) \hookrightarrow \mathbb{C}$ (For example, let λ = something)

$\bar{\eta} \in \dfrac{\text{D.S.K.}}{\text{exact}}$. Consider: $H_1(E_\lambda, \mathbb{Z})$.

$H^1_{DR}(E_\lambda) \longrightarrow \mathrm{Hom}(H_1(E_\lambda, \mathbb{Z}), \mathbb{C})$

given by $\bar{\eta}$: for $\gamma \in H_1(E_\lambda, \mathbb{Z})$

~~$\int_\gamma \bar{\eta} \in \mathbb{C}$~~. $\bar{\eta} \mapsto \left(\gamma \mapsto \int_\gamma \bar{\eta} \boxed{\in \mathbb{C}}\right)$.

This map is isomorphism.

Consider

$$\omega = \frac{dx}{y} \qquad \eta = \frac{x\,dx}{y}.$$

Look at $\bar{\omega} \in H^1_{DR}(E_\lambda)$.

$\bar{\omega}$, $D_\lambda(\bar{\omega})$, $D_\lambda^2(\bar{\omega})$.

Note that $H^1_{DR}(E_\lambda)$ is 2-dim vector space over $\mathbb{Q}(\lambda)$.

They satisfies:

$$\lambda(1-\lambda)\, D_\lambda^2(\omega) + (1-2\lambda)\, D_\lambda(\omega) - \frac{1}{4}\omega$$

$$= d\left(\frac{1}{4}\frac{-y}{(x-\lambda)^2}\right)$$

$$\lambda(1-\lambda)f'' + (1-2\lambda)f' - \frac{1}{4}f = 0$$

Classical D.E.

$$F\left(\frac{1}{2}, \frac{1}{2}, 1, \lambda\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}\lambda^n.$$

where $\left(\frac{1}{2}\right)_n = \frac{1}{2}\cdot\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n-1\right)$.

About Hilbert's problem.

Given finite number of points : in $\mathbb{C}$.

$$\rho : \pi(\mathbb{C} - \{p_i\text{'s}\}) \longrightarrow Gl_n(\mathbb{C}).$$

Q $\exists$? D.E., with coeff. ......

Riemann-Hilbert Problem.