9月24日 *: I add it. /5-1

General definition of differential operators:

K: commutative ring,

R: commutative K-algebra.

$D_K(R)$: the ring of K-linear Operators.

$\delta: R \to R$

$D_K(R)$ is a subring of $End_K(R)$.

$D_K(R) = \bigcup_{n=0}^{\infty} D_K^n(R)$.

$D_K^n(R)$ is the left R-module of K-linear diff. operators of order $\le n$.

Def: ① $\delta \in D_K^0(R)$ iff δ is the multiplication by an element of R.

② $\delta \in D_K^n(R)$ if it is K-linear and $[\delta, a] \in D_K^{n-1}(R)$ for any $a \in R$.

Corollary:

$D_K^n(R)$ is a left R-module.

Proof, $r \in R$, $\delta \in D_K^n(R)$, $a \in R$, then,

$[r\delta, a] = r[\delta, a] + [r, a]\delta = r[\delta, a] \in D_K^{n-1}(R)$

by the inductive assumption on $n-1$.

*(So of course it is also a right R-module)

Proposition: $D_K(R)$ is a ring.

($D_K(R) \subset \mathrm{End}_K(R)$, so multi. is already defined).

More precisely, $\delta' \in D_K^m(R)$ $\delta \in D_K^n(R)$, then $\delta \circ \delta' \in D_K^{m+n}(R)$.

Pf: Induction on $m+n$.

$[\delta\delta', a] = \delta[\delta', a] + [\delta, a]\delta'$. (*when $m=0$ or $n=0$, reduced to Module case)

*Up to now, nothing tells you there exists a diff operator. (except multi.)

Alg. def. is always like this.

5-3

<u>Recall</u>: K-linear derivation is a K-linear Map $\delta: R \to R$, s.t.

$\delta(ab) = a\,\delta(b) + \delta(a)\,b$. (e.g. $[\cdot, c]$).

$\mathrm{Der}_K(R)$ is naturally a left R-module.

(Messing: Also a K-lie alg.)

~~Property~~ Lemma: $\mathrm{Der}_K(R) \oplus R = D^1_K(R)$

<u>Pf</u>: ① $\delta \in \mathrm{Der}_K(R)$

<u>What we need</u>: $[\delta, a] \in D^0_K(R)$.

$[\delta, a]\,r = \delta(ar) - a\,\delta(r) = \delta(a)\cdot r$ This proves "$\subseteq$"

② If $\delta \in D^1_K(R)$, Set $\delta' = \delta - \delta(1)$

*(Taking action on "1" gives a projection)

By def. of D^1_K, $[[\delta', a], b] \equiv 0$, $\forall a, b \in R$.

Apply to "1", we have:

$\delta'(ab) - a\,\delta'(b) - b\,\delta'(a) \equiv 0$. ↗ $\delta'(ab) = a\,\delta'(b) + b\,\delta'(a)$!!

* very good!

Prop: Every K-linear derivation of $R = K[x_1, \dots, x_n]$ is of the form $\delta = \sum f_i d_i$, $f_i \in R$, $d_i = \frac{\partial}{\partial x_i}$. 5-4

*(Now things begin to make sense)

Pf: $\delta \in \mathrm{Der}_K(K[x_1, \dots, x_n])$, then

$$\delta(x_1^{s_1} \cdots x_n^{s_n}) \overset{*}{=} \underset{\delta(x_1^{s_1})x_2^{s_2}\cdots x_n^{s_n}}{\underset{\|}{\square}} + \square + \square + \square + \square \cdots$$

$$\delta x_i^{s_i} = s_i \cdot x_i^{s_i - 1} \cdot \delta(x_i) = \delta(x_i) \cdot d_i(x_i^{s_i}).$$

*($\in R$, so commutes with x_i)

So the whole thing $= \delta(x_1)d_1 + \cdots + \delta(x_n)d_n$ when applying to $x_1^{s_1} \cdots x_n^{s_n}$. Let s_i vary, so ... !!

*($\delta(x_i) = f_i$ is a good way to find f_i.
This method can apply to higher degrees.)

Thm: $A_n(K)$ is the ring of K-linear diff. operators of R (provided K contains a field of char 0).

Of course things are similar ~~in~~ for $D^s_k(k[x_1,\dots,x_n])$

<u>Exer</u>: $R = k[t^x, t^2]$,

$D_k(R)$ is not generated by $\mathrm{Der}_k(R)$.

<u>Fact</u>: If R is the coor ring of a non-singular variety over k with char 0, then

$D_k(R)$ ARE generated by $\mathrm{Der}_k(R)$.

<u>Fact</u>: $D_k(k[x_1,\dots,x_n])$ is generated by

$\left\{ \frac{1}{t!} \frac{\partial^t}{\partial x_i^t} \right\}$. $\forall\, k$!!! ~~[illegible]~~

(e.g. $k = \mathbb{Z}$, but not $\mathbb{F}_p$).

Same for $k[[x_1, \dots, x_n]]$. ($D_k \to$ continuous in this case.)