$R = k[X_1, \cdots, X_n]$ k: field.

$d_i = \dfrac{\partial}{\partial X_i}$, : $R \to R$.

If $\mathrm{char}(k) = p > 0$, then $d_i^p = 0$. (easy check)

$\boxed{\text{Blanket assumption: } \mathrm{char}\, k = 0}$ (so $d_i^t \neq 0$, $\forall t$)

$X_i : R \to R$, (multiplied by X_i).

Def: $A_n(k)$ is the k-subalgebra of $\mathrm{End}_k(R)$ generated by $\{X_1 \sim X_n, d_1 \sim d_n\}$.

$A_n(k)$ is "the Weyl algebra".

Relations: ① $[X_i, X_j] = 0$.

② $[d_i, d_j] = 0$.

③ $[d_i, X_j] = 0$ if $i \neq j$.

④ $[d_i, X_i] = 1$.

<u>Prop</u>: monomials $\{x_1^{i_1}\cdots x_n^{i_n} d_1^{j_1}\cdots d_n^{j_n}\}$ are a k-basis of $A_n(k)$.

(The relations imply that $\{..\}$ is the generators, For linear independence, ~~Consider the degrees of x_i's when it acts on $x_1^{c_1}\cdots x_n^{c_n}$~~). Consider the action on one of the highest degree elements of the relation.)

Pick a new set of generators of R:

$y_1, \cdots, y_n, \quad y_i = f_i(x_1, \cdots, x_n)$ s.t.

$k[x_1 \cdots, x_n] = k[y_1, \cdots, y_n] = R$

~~Then~~ Let $\bar{d_i} = \frac{\partial}{\partial y_i} : R \to R$ (w.r.t. $y_1, \cdots, y_n$).

Then: $\alpha : A_n(k) \to A_n(k)$ sending x_i to y_i, d_i to $\bar{d_i}$

is an automorphism of $A_n(k)$.

Degree of $\delta \in A_n(k)$ is $\max(\sum i_? + \sum j_?)$.

<u>Fact</u>: $(x_1^{i_1} \cdots x_n^{i_n} \cdot d_1^{j_1} \cdots d_n^{j_n}) \cdot (x_?^{i'_?} d_?^{j'_?})$ has leading term $x^{i+i'} d^{j+j'}$.

with the rest of ~~tem~~ terms' degree at most (Top − 2).

<u>Thm</u>: ① $\deg(\delta+\delta') \le \max \cancel{(\delta+\delta')} \{\deg\delta, \deg\delta'\}$

② $\deg(\delta_1 \cdot \delta_2) = \deg\delta_1 + \deg\delta_2$. (need some check).

③ $\deg[\delta, \delta'] \le \deg\delta + \deg\delta' - 2$.

<u>Coro</u>: $A_n(k)$ is a domain.

<u>thm</u>: $A_n(k)$ is a simple ring.

(no nontrival two-side ideals).

<u>Pf</u>: $\forall\ \delta \in A_n(k)$, multipling on left or right, you will get the whole ring.

<u>Ex</u>: $[d_i^t, x_i] = t \cdot d_i^{t-1}$, $[d_i, x_i^t] = t \cdot x_i^{t-1}$.

<u>Then</u>, we use the induction on $\deg(\delta)$.

Consider $[\delta, x_i]$ or $[\delta, d_i]$. some discussions ...

Coro: Every endomorphism of $A_n(k)$ is injective.

Fact: Every left ideal is generated by 2 elements.

Assume $\mathrm{char}(k) = p > 0$.

Fact: Set $R_n = k[X_1, \cdots, X_n, d_1, \cdots, d_n]$ subject to relations (d_i's ~~is~~ are not differentials).

R_n is not simple.

Ex: $R_1 = k[X_1, d_1]$

Claim: $[d_1, X_1^p] = 0$.

So (X_1^p) is a 2-side ideal.

$A_n(k) = R\langle \frac{1}{t!} \frac{\partial}{dx_i^t} \rangle$ is also considered.