Noetherian Rings & Modules (chapter 8).

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"$\mathcal{N}$ = Noetherian" "F.G. = finitely generated".

$$S_n = gr^B(A_n(\cancel{k})) \cong k[Y_1, \cdots, Y_{2n}].$$

By Hilbert, Noetherian!

By this, we conclude that A_n is a left $\mathcal{N}$-ring.

General thm:

M: left A_n-module, Γ is the filtration w.r.t. B.
↑ fil. on A_n.

~~then~~ if $gr^\Gamma(M)$ is $\mathcal{N}$ ~~,~~ then M is $\mathcal{N}$.

Remark. F.G. Modules may not be $\mathcal{N}$.

$gr^\Gamma(M)$ is F.G. $\Rightarrow$ M is.

but $\Leftarrow$ not true!

Γ is "good" if $gr^{\Gamma} M$ is f.g.

(M is f.g. because "$\Rightarrow$" is right)

- Good filtration criterion $\longrightarrow$ examples of filtrations which are not good.

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① Noetherian Modules.

R ring.; M : left ~~Mom~~ R module.

M is $\mathcal{N}$ if all its submodules are f.g.

<u>Thm 1.1</u>: $\left\{ \begin{array}{l} \text{For every } \overset{\text{acc}}{\cancel{\text{ass}}}\text{ending chain} \\ N_1 \subset \cdots \quad \text{of } \cancel{\text{so}} \text{ submodules of } M. \\ \exists\ k > 0, \text{ s.t. } N_i = N_k \text{ when } i \geq k. \end{array} \right\} \Leftrightarrow \{M \text{ is } \mathcal{N}\}$

$\Leftrightarrow \left\{ \begin{array}{l} \text{Every Set } S \text{ of Submodules of } M \\ \text{has a max. element} \end{array} \right\}$

$\uparrow$

RM: Without assuming a weak form of Axiom of Choice.

"Max element" $\Longrightarrow$ previous two. (by Messing)

$\not\Longleftarrow$

Prop 1.3 M: left R-module, $N \subseteq M$ submod.

① M is $\mathcal{N}$ iff both M/N and N ~~is~~ are.

② $N' \subset M$, subm., ~~suppose~~ $M = N + N'$.

~~then~~ If N, N' are $\mathcal{N}$, then so is M.

So, homomorphic image of $\mathcal{N}$ module is $\mathcal{N}$-module

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§2 Noetherian Rings.

Def: R called a left Noetherian Ring if it is $\mathcal{N}$ as a left module of it self, or equivantly, all left ~~mo~~ ideals are F.G.

Recall Hilbert basis thm:

R: commutative, then $R[x]$ is $\mathcal{N}$. if R is

In particular, $R = k$, a field, it is true.

<u>Thm 2.3</u>. M is a left A_n-module with filtration $\Gamma_\bullet$ w.r.t. B.

If $gr^\Gamma M$ is a $\mathcal{N}$ S_n-module, then M is $\mathcal{N}$.

<u>proof</u>. Let N by a submodule of M.

Let $\Gamma_i' = \Gamma_i \cap N$, the restriction of fil. from M to N.

Then $gr^{\Gamma'} N$ is subm. of $gr^\Gamma M$, so

$gr^{\Gamma'} N$ is F.G.

Let m be the max. degree of a finite set of generators of $gr^{\Gamma'} N$. then we claim: N is generated by elements in Γ_m'. Note that $\dim \Gamma_m' < \infty$ (by convention), so N is f.g.

* I think, the assumption $\dim \Gamma'_m < \infty$ is not neccessary. The generators of $gr^{\Gamma'} N$ can be lifeed to be ~~a~~ generators of N. Messing agrees.

Last two pages of Cartan Eilenberg ~~Steenrod~~ «homological alg.» Chapter 1

example of lefe Noetherian ring but not right.

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§ 3. Good filtrations

M. left A_n-module.

$gr^{\Gamma} M$ is F.G. $\Rightarrow$ M is

but $\Leftarrow$ not true.

Def: $gr^{\Gamma} M$ is f.g. then call Γ "good".

RM: M is F.G. $\Rightarrow \exists$ "good" filtration.

let $\langle u_1 \dots u_s \rangle$ be generators, let $\Gamma_k = \sum B_k u_i$

then Γ is good.

Prop 3.1. a good filtration Criterior.

M left A_n-module Γ.

Γ is good if $\exists k_0$, s.t $\Gamma_{i+k} = B_i \Gamma_k$ for all i and all $k \geq k_0$

$\Leftarrow$ easy, use Γ_{k_0} to create.

$\Rightarrow$ Let $gr^{\Gamma}(M)$ be F.G. with $\langle u_j \rangle$.

Each u_j has degree k_j (well-defined!)

then $k_0 = \max\{k_1, \dots, k_s\}$ would work.

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* So gr^{Γ} is a good functor to investigate F.G. ~~Noetherian~~. It preserves the structure and connects to commutative ring.

$\exists$ Not good filtrations. Be careful!

Prop 3.2. M. left A_n-mod. Γ. Ω are fils.

1. Γ is good. $\Rightarrow \exists\, k_1$, s.t. $\Gamma_j \subseteq \cancel{\Gamma_{j+k}}\ \Omega_{j+k_1}$. $\forall j$

2. Γ, Ω good $\Rightarrow \exists\, k_2$. $\Omega_{j-k_2} \subseteq \Gamma_j \subseteq \Omega_{j+k_2}$.