Modules over the Weyl alg. & differential equations

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General setup:

$R \curvearrowright M^{\neq 0}$ — All below is "Left"

Comm. (left) module

① We say $M^{\neq 0}$ is irred (simple) if no nontrivial submodules.

- $u \in M$, $\mathrm{Ann}_R(u) = \{r \in R \mid ru = 0\}$
  it is a left ideal in R.

- u is torsion if $\mathrm{ann}_R(u) \neq \cancel{\emptyset}\,(0)$.
  M is torsion if all elements are torsion.

Lem. If M is irred. $u \in M$,

then $M \simeq R/\cancel{\mathrm{Ann}_u(R)}\, \mathrm{ann}_R(u)$

proof: $r \longmapsto ru$

$R \longrightarrow M$.

Hom of R-modules.

Lem. If R is not a div. ring,

then M is torsion.

Pf: If not, $\exists\, u \in M$, $u \neq 0$, $\mathrm{ann}_R(u) = \{0\}$.

So $\cancel{r \mapsto ru}$

$M \simeq R/\mathrm{ann}_R(u) = R$.

But if R is irred. as a left R-module,

So every left principle ideal is R, so R is

div. ring! (commutative).

Example:

(a). $A_n \curvearrowright k[x_1 \sim x_n]$.

Prop: $k[x]$ is irred, an is iso. to. $A_n/(\partial_1, \dots, \partial_n)$.

If $f \in k[x]$, $f \neq 0$, we show f generates $k[x]$.

x^α be ~~the~~ a max. deg term. of f.

Take $\partial^\alpha x^\alpha$ where $\partial^\alpha = \partial_1^{i_1} \cdots \partial_n^{i_n}$

then $\partial^\alpha x^\alpha = c$. so $1 \in A_n f$.

So $k[x]$ is irred.

Take: $A_n \to k[x]$
$a \mapsto a \cdot$ (Acting on 1)

$\mathrm{ann}_{A_n}(1) = (\partial_1, \dots, \partial_n)$. (Not hard to check)

~~Obviously. $(\partial_1, \dots, \partial_n) \in \mathrm{ann}_{A_n}(1)$.~~

More generally, can consider.

$$A_n \big/ (\partial_1 - g_1, \dots, \partial_n - g_n) \simeq \underline{k[x]}.$$

∂_i replaced by $\underline{k[x]}$ things in.

$g_i \in k[x]$.

Remark. if $a = f + c_1\partial_1 + \cdots + c_n\partial_n$,

$f \in k[x]$, $c_i \in A_n$,

then $a = f + \Sigma c_i g_i$.

~~*(how to play with this module?)~~

<u>Another example</u>: $k[\partial] = A_n \big/ (x_1, \dots, x_n)$.

$= k[\partial_1, \dots, \partial_n]$

Later we will interpret this as the "Fourier transform" of $k[x]$.

~~On Generally~~

General setup again:

$R \curvearrowright M$,

$\sigma : R \to R$ automorphism.

Let $M^{\sigma} = M$ but R acts by

$r \circ m = \sigma(r)^{\sigma}(m)$.

In fact: $M^{\sigma} \simeq R \otimes_{\sigma^{-1}} M$.

(Here, R is bi-module.)

Prop: ① M_{σ} irred. $\Leftrightarrow$ M is.

② M_{σ} torsion $\Leftrightarrow$ M torsion

③ N submodule $\Leftrightarrow$ $(M/N)_{\sigma} \simeq M_{\sigma}/N_{\sigma}$.

④ $(R/J)_{\sigma} = R/\sigma^{-1}(J)$

$$F: A_n \xrightarrow{\sim} A_n.$$
$$x_i \longmapsto \partial_i$$
$$\partial_i \longmapsto -x_i.$$

(preserve the relations $[\partial_i, x_i] = 1$).

Now consider

$$\begin{array}{ccc} k[x]_F & \simeq & k[\partial] \\ \wr\wr & & \wr\wr \\ A_n/(\partial_1, \dots, \partial_n) & & A_n/(x_1, \dots, x_n). \end{array}$$

Another Example:

$$\sigma: A_n \longrightarrow A_n$$
$$x_i \longmapsto x_i$$
$$\partial_i \longmapsto \partial_i + g_i$$

$$k[x]_\sigma \simeq A_n/\sigma^{-1}(\partial_1, \dots, \partial_n).$$

This means A_n acts by σ.

(without σ, A_n acts by "id")

Create. distinct (Mutually non-iso.)

irred. Modules:

$\forall r \in \mathbb{Z}_{>0}$, let

$\sigma_r : A_n \longrightarrow A_n$.

$x_i \longmapsto x_i$

$\partial_i \longmapsto \partial_i - x_i^r$.

<u>then</u>: $\{k[x]_{\sigma_r}\}$ for $r \in \mathbb{Z}_{>0}$ form an

infinite set of ~~dis~~ distinct irred. Modules.

Holomorphic functions: $\underline{k = \mathbb{C}}$.

$U \underset{\text{open}}{\subset} \mathbb{C}$, $\mathcal{H}(U) = \{\text{holo. functions on } U\}$.

$A_1(\mathbb{C})$ acts on $\underline{\mathcal{H}(U)}$.

Huge space,

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$H(\mathcal{U})$ is not torsion:

$h = e^{e^z}$ is not ...

Fact, $\forall\, m \in \mathbb{Z}_{>0}$, $\exists$ polynomial $F_m(z)$.

$$\partial^m h = F_m(e^z)\, h.$$

(Consider degree of e^z)

<u>Consider</u>: linear diff. system with polynomial equations:

$$P_i f = 0, \quad P_i \in A_n.$$

$f \in$ any A_n-modules

f is a polynomial solution if $f \in K[x]$.

Want M, an A_n-module.

$\mathrm{Hom}_{A_n}(M, S)$ ~~No A_n structure on it~~

$\cong$ $\{$ solution of $(P_i f)$ in $S \}$ as K. Vector space's.

<u>Ans</u>: $M = A_n / (P_i\text{'s})$

For each ~~$f \in \mathrm{Hom}_{A_n}$~~ solution f,

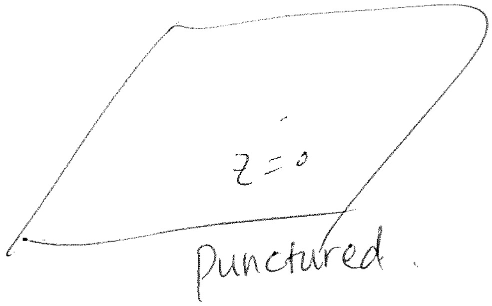
define $A_n \xrightarrow{\sigma_f} S$

$1 \mapsto f$.

$\forall p \in (P_1, \dots, P_m)$, $p \mapsto pf = 0$.

$\overline{\sigma_f} : A_n / (P_i\text{'s}) \to S$.

Micro functions $\underline{K=\mathbb{C}}$.



$\varepsilon>0$, $D(\varepsilon)=\{z \mid |z|<\varepsilon\}$

$D'(\varepsilon)=D(\varepsilon)\setminus\{0\}$.

$\tilde{D}(\varepsilon)$ = univ. cover by $\{\tau\in\mathbb{C} \mid \mathrm{Re}(\tau)<\log\varepsilon\}$

$$\begin{tikzcd}
\tau\in\tilde{D}(\varepsilon) \arrow[d, "\pi(\tau)=e^{\tau}=z"] \arrow[dr, bend left, "\bar{\pi}"] & \\
z\in D'(\varepsilon) \arrow[r, hook] & D(\varepsilon)
\end{tikzcd}$$

then $\mathcal{H}(D(\varepsilon)) \hookrightarrow \mathcal{H}(\tilde{D}(\varepsilon))$

$A_1(\mathbb{C})$ acts on $D(\varepsilon)$. What about $\tilde{D}(\varepsilon)$.

~~A~~ $A_1(\mathbb{C})$ acts on $\tilde{D}(\varepsilon)$ by $\frac{\partial}{\partial z}$, not $\frac{\partial}{\partial \tau}$.

$$M_\varepsilon = \mathcal{H}(\tilde{D}(\varepsilon)) / \bar{\pi}^*(\mathcal{H}(D(\varepsilon))) = \bar{\pi}_*(\mathcal{H}(\tilde{D}(\varepsilon)))/\mathcal{H}(D(\varepsilon))$$

$M = \varinjlim_{\varepsilon} M_\varepsilon$. Microfunctions.

Examples: $\mathrm{can}: H(\tilde{D}(\varepsilon)) \to M_\varepsilon \to M.$

$$\delta = \mathrm{can}\left(\frac{1}{2\pi i} e^{-z}\right).$$ Direc delta.

$$z\delta = \mathrm{can}\left(0 \frac{1}{2\pi i}\right) = 0$$ since $\frac{1}{2\pi i}$ can be extended.

$$Y = \mathrm{can}\left(\frac{1}{2\pi i} \tau\right) = \mathrm{can}\left(\pi^*\left(\frac{1}{2\pi i} \cdot \log(z)\right)\right)$$

$$\partial Y = \mathrm{can}\left(\pi^*\left(\frac{1}{2\pi i} \cdot \frac{1}{z}\right)\right) = \delta$$

$$\partial Y = \mathrm{can}\left(\frac{1}{2\pi i} e^{-z}\right) = \delta.$$