§1 : D-modules with support

For a module M over a commutative ring R, and an ideal $I \subset R$

we may define $\Gamma_{V(I)}(M) := \left\{ \begin{array}{l} u \in M \text{ s.t.} \\ \exists n \geq 0 \quad I^n u = 0 \end{array} \right\}$

For our set-up, we have:

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & X \times Y \\ \| & & \| \\ \mathbb{A}^n_k & & \mathbb{A}^m_k \end{array} \qquad \iota(X) = H$$

For an A_{n+m}-module M, define $\Gamma_H(M)$ by viewing M as $K[X \times Y]$-module (or as $K[Y]$-mod.)

Lemma: $\Gamma_H(M) \subset M$ is an A_{n+m}-submodule

pf: All clear except for action by ∂_{y_j}.

If $u \in \Gamma_H(M)$, $y_j^k u = 0$ (or more generally $y^\alpha u = 0$) then $y^{\alpha + e_j} \partial_{y_j} u =$

$$\partial_{y_j} y^{\alpha + e_j} u - (\alpha_j + 1) y^\alpha u$$

$$= 0.$$

so $\partial_{y_j} u \in \Gamma_H(M)$.

If $\Gamma_H(M) = M$ then we say M has support on H.

Given any A_{n+m}-module, $\mathrm{Ker}_M(y) = \{ u \in M : (y) u = 0 \}$, where $(y) = (y_1, \dots, y_m)$

$$\mathrm{Ker}_M(y) \subset \Gamma_H(M)$$

this is only A_n-module not an A_{n+m}-module.

Let $M_0 := \mathrm{Ker}_M(y)$. Then $A_{n+m}(M_0) \subset \Gamma_H(M)$.

Lemma (1) $\forall$ multidx α, β $(\in \mathbb{Z}^m_{\geq 0})$

$$\partial_y^\alpha : \partial_y^\beta M_0 \longrightarrow \partial_y^{\alpha+\beta} M_0 \quad \text{is injective}$$

(2) $$y^\alpha (A_{n+m} M_0) = A_{n+m} M_0 \quad \forall \alpha \in \mathbb{Z}^m_{\geq 0}$$

(3) $$A_{n+m} M_0 = \bigoplus_{\alpha \in \mathbb{Z}^m_{\geq 0}} \partial_y^\alpha M_0$$

Pf: $$y_j \partial_{y_j}^{\alpha_j} = \underbrace{\partial_{y_j}^{\alpha_j} y_j}_{\text{acts by } 0 \text{ on } M_0} - \alpha_j \partial_{y_j}^{\alpha_j - 1} \Rightarrow$$ if $u \in M_0$,

$$y^\alpha \partial_y^\beta u = (-1)^{|\alpha|} c_\alpha^\beta \partial_y^{\beta-\alpha} u$$

w/ $\partial_y^{\beta-\alpha} = 0$ if $\alpha_j > \beta_j$ for any j

$$c_\alpha^\beta = \prod_j \beta_j(\beta_j - 1)\cdots(\beta_j - \alpha_j + 1)$$

This gives (1) – (3).

Lemma: Suppose $m = 1$, then as A_{n+1}-modules, we have (for $i : X \to X \times Y$)

$$i_*(M_0) \simeq M_0 \boxtimes K \simeq A_{n+1} M_0 = \Gamma_H(M)$$

last time

the isom. follows from previous lemma

need to show equality here, not just containment $\subseteq$.

We must show $A_{n+1} M_0 = \Gamma_H(M)$

Suppose $u \in \Gamma_H(M)$ and $y^\alpha u = 0$. Want $u \in A_{n+1} M_0$.

For $\alpha = 1$: $yu = 0 \Rightarrow u \in M_0$ by def'n.

Assume $\alpha > 1$, true for all smaller α by induction.

$$0 = \partial_y y^\alpha u = y^\alpha \partial_y u + \alpha y^{\alpha-1} u = y^{\alpha-1}(y \partial_y u + \alpha u)$$

$$\Rightarrow y \partial_y u + \alpha u \in A_{n+1} M_0 \quad \text{by induction.}$$

On other hand, $0 = y^\alpha u = y^{\alpha-1}(yu) \Rightarrow yu \in A_{n+1} M_0$

$\Rightarrow \quad \partial_y yu \in A_{n+1} M_0$

$\Rightarrow \quad \alpha u + y\partial_y u - \partial_y yu = (\alpha - 1) u \in A_{n+1} M_0$.

§2 We can reformulate the last lemma as

Thm (Kashiwara's equivalence)

$$\iota : X \hookrightarrow X \times Y \qquad (Y = \mathbb{A}_k^m)$$

$$x \longmapsto (x, 0)$$

then ι_* induces an equivalence of categories $\mathcal{M}^n = (A_n\text{-modules})$ and $\mathcal{M}^{n+m}(H) \overset{\text{def}}{=} (A_{n+m}\text{-mod's with support on } H = \iota(X))$

$$\mathcal{M}^n(X) \longrightarrow \mathcal{M}^{n+m}(H)$$

$$N \longmapsto \iota_* N$$

$$M_0 = \mathrm{Ker}_M(y) \longleftarrow M$$

pf: When $m > 1$, $\iota : X \hookrightarrow X \times Y$ can be factored as

$$X \xrightarrow{\iota_1} X \times Z \xrightarrow{\iota_2} (X \times Z) \times W$$

$$Z = \mathbb{A}_k^{m-1}, \quad W = \mathbb{A}_k^1$$

First $\iota_* N = (\iota_2)_* (\iota_1)_* N$. Moreover

$$\mathrm{Ker}_{(\mathrm{Ker}_M \iota_2(X \times Z))}(\iota_1(X)) = \{u \in M : y_1 u = 0, \dots, y_m u = 0\} = \mathrm{ker}_M(\iota(X))$$

So we have reduced to case $m = 1$. By previous lemma, $\iota_* M_0 \cong A_{n+1} M = \Gamma_H(M) = M$.

If $M \to M'$ is a morphism of A_{n+1}-modules, this induces

map $M_0 \to M_0'$, which induces $M = A_{n+1} M_0 \to A_{n+1} M_0' = M'$.

On other hand, since only elts of $K[\partial_y]$ annihilated by y are the constants in K, we have:

$$\ker_{\iota_*(M_0)}(y) \simeq \ker_{M_0 \boxtimes K[\partial_y]}(y) = M_0$$

If $M_0 \to M_0'$ then $\iota_* M_0 \simeq M_0 \boxtimes K[\partial_y] \to \iota_* M_0' \simeq M_0' \boxtimes K[\partial_y]$

induces the same morphism $M_0 \to M_0'$. //

§ 3 If M holonomic, then so is $\iota^* M$. $(*)$

By induction, we may assume $m = 1$.

Lemma 1 : If M is A_{n+1}-module and $M' = M / \Gamma_H(M)$, $H = \iota(X)$,

then $\iota^*(M) \simeq \iota^*(M')$

pf : b/c $\iota^* M \simeq M / yM$; but we know $y \cdot \underbrace{\Gamma_H(M)}_{y(A_{n+1} M_0) = A_{n+1} M_0 .} = \Gamma_H(M)$.

Lemma 2 : $(*)$ is true if $\Gamma_H(M) = \{0\}$.

pf : Take Γ: any good filtration on M. (for Bernstein filtration on A_{n+1})

Take $\Omega_j = (\Gamma_j + yM) / yM$ induced by Γ_j (for A_n)

Since $\Omega_j \simeq \Gamma_j / \Gamma_j \cap yM$ and $y \Gamma_{j-1} \subset \Gamma_j \cap yM \Rightarrow \dim_K(\Omega_j)$

$\leq \dim_K(\Gamma_j)$

$- \dim_K(y\Gamma_{j-1})$

Since $\Gamma_H(M) = 0$, left multiplication by y is injective

$\Rightarrow \quad \dim_k(y\Gamma_{j-1}) = \dim_k(\Gamma_{j-1}) \Rightarrow \dim_k(\Omega_j) \leq \dim(\Gamma_j) - \dim(\Gamma_{j-1})$

$\Rightarrow$ For $j \gg 0$, $\dim(\Omega_j) \leq \chi(j, M, \Gamma)$

$$- \chi(j-1, M, \Gamma)$$

same leading term

$\frac{m(M)}{n!} j^n$ (via holonomy of M here)

$\Rightarrow \exists\, c$ s.t.

$$\dim_k(\Omega_j) \leq \frac{m(M)}{n!} j^{n-1} + c \cdot (j+1)^{n-2}$$

$\Rightarrow$ Lemma 10.3.1 says that M/yM is also holonomic (as A_n-module)

with $m(M/yM) \leq m(M)$. //

If $F : X \to Y$ polynomial, then $F^*(\text{holon.})$ is holon.

$F_*(\text{holon.})$ is holon.

via factorization of F into ι, G, π.

<u>Remark</u>: $\pi : X \times Y \to Y$. We want $\pi_* N$ to be holonomic if N holon.

$\pi_* N = N/(\partial_x)N$ is the Fourier transform of

$(\iota')^* N' \simeq N'/(x_i)N'$ where N' = Fourier transform of N

$$\iota' : Y \to X \times Y$$
$$y \mapsto (0, y)$$