K : field of char. 0

$X = \mathbb{A}^n_K$ ($= K^n$ in book)

$Y = \mathbb{A}^m_K$

Let A_n, A_m be their respective Weyl algebras

$F : X \to Y$ "polynomial map"

i.e. $F_\# : K[Y] = K[y_1, \dots, y_m] \to K[X] = K[x_1, \dots, x_n]$

$y_j \longmapsto F_j$ (polys.)

$X \times Y = \mathbb{A}^{n+m}_K$ (over $\mathrm{Spec}(K)$) with assoc. Weyl algebra $A_{n+m} = A_n \otimes_K A_m$

Given N, M respective A_n, A_m - module, we have $N \boxtimes M$ ($= N \hat{\otimes} M$ in book)

F · factors as composition :

$$X \overset{i}{\hookrightarrow} X \times Y \xrightarrow{G} X \times Y \overset{\pi}{\twoheadrightarrow} Y$$

$x \longmapsto (x, 0)$

$(x, y) \longmapsto (x, y + F(x))$

$(x, y) \longmapsto y$

Goals : ① define inverse / direct images (as functors)

Given $F : X \to Y$ then want $\begin{pmatrix} D\text{-modules}/X \\ = A_n\text{-modules} \end{pmatrix} \underset{F_*}{\overset{F^*}{\rightleftarrows}} \begin{pmatrix} D\text{-modules}/Y \\ = A_m\text{-modules} \end{pmatrix}$

② study these under 'standard embeddings" like i, "automorphisms" like G, "projections" like π, "compositions"

③ Explain a special case of Kashiwara's equivalence :

$\begin{pmatrix} D\text{-modules}/X \\ A_n\text{-module} \end{pmatrix} \xrightarrow{i_*} \begin{pmatrix} D\text{-modules}/X \times Y \\ \text{with support on} \\ i(X) \subset X \times Y \end{pmatrix}$ is an equivalence of categories.

this to be defined later

Remark: For general closed embeddings of smooth varieties, if is étale locally given by $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$ like this 1.

Side remark: If X is singular, so locally embedded in smooth varieties (e.g. quasi-proj.) we can define $\mathcal{D}$-modules with supports without needing to use smoothness of X (makes sense because of Kashiwara's equivalence)

§ 2 Transfer bimodules

(A) $F: X \to Y$, then $D_{X \xrightarrow{(F)} Y} := K[X] \underset{K[Y]}{\otimes} A_m$ an A_n-A_m-bimod. ($K[X] = K[x_1, \dots, x_n]$)

but left action of A_n has to be explained.

For every $h \in K[X]$, $u \in A_m$, then define:

$$x_i(h \otimes u) = (x_i h) \otimes u \quad \text{(usual } K[X]\text{-action)}$$

$$\partial_{x_i}(h \otimes u) = (\partial_{x_i} h) \otimes u + \sum_{j=1}^{m} h(\partial_{x_i} F_j) \otimes (\partial_{y_j} u)$$

(rt. A_m action is on right-hand component A_m.)

where $y_j \mapsto F_j$ under $F_\#$

"push forward of ∂_{x_i}"

Need to check that

$$\partial_{x_i}(h \otimes y_{j_0} u) = \partial_{x_i}(h F_{j_0} \otimes u)$$

(B) transposition: $\exists$ anti-autom. $\tau: A_m \xrightarrow{\sim} A_n^{opp}$

$$\tau(a)\tau(b) = \tau(ba) \quad \tau^2 = \mathrm{Id}.$$

switches all operations $M \mapsto M^t$ from left to right modules.

$$h\,\partial^\alpha \longmapsto (-1)^{|\alpha|} \partial^\alpha h$$

(c) $\cdot\, D_{Y \leftarrow X} := (D_{X \to Y})^t$ ← twist by τ

this takes A_m-A_n-bimods to A_n-A_m-bimods where L-R-bimod always means L acts on left. R acts on rt.

(D) $\begin{matrix} X & \xrightarrow{F} & Y & \xrightarrow{G} & Z \\ \parallel & & \parallel & & \parallel \\ \mathbb{A}^n & & \mathbb{A}^m & & \mathbb{A}^l \end{matrix}$, then $D_{X \to Y} \otimes_{A_m} D_{Y \to Z} \simeq D_{X \to Z}$ as A_n-A_l-bimod

and then by transposition: $D_{Z \leftarrow Y} \otimes_{A_m} D_{Y \leftarrow X} \simeq D_{Z \leftarrow X}$ as A_l-A_n-bimod.

§ 3 $F^* M = D_{X \to Y} \otimes_{A_m} M$ for any A_m-module M (for Y)

$F_* N = D_{Y \leftarrow X} \otimes_{A_n} N$ for any A_n-module N (for X)

Facts (A): For standard embedding $\iota : X \to X \times Y$
$x \mapsto (x, 0)$

$$D_{X \to X \times Y} \simeq K[X] \otimes_{K[X \times Y]} A_{n+m}$$

$$\simeq A_{n+m} / (y_1 \cdots y_m) A_{n+m} =: A_{n+m} / (y) A_{n+m}$$

$$D_{X \times Y \leftarrow X} \simeq \left(A_{n+m} / (y) A_{n+m} \right)^t = A_{n+m} / A_{n+m}(y) \simeq A_n \otimes_K K[\partial_y]$$

$\Rightarrow$ $\iota^* M = M / (y) M$

$\iota_* N = N \boxtimes K[\partial_y]$

$\Rightarrow$ if N is fin-gen as A_{n+m}-mod, then

① $\iota_* N$ f.g. as A_{n+m}-mod

② $d(\iota_* N) = m + d(N)$

(so N holonomic $\Rightarrow$ so is $\iota_* N$)

③ $m(\iota_* N) \leq m(N)$.

However, for general fin. gen. M,
$\iota^* M$ is not fin. gen.

If M holonomic, it's true that
$\iota^* M$ holonomic. (needs proof)

(B) for projections $D_{X\times Y\to Y} \simeq K[X\times Y] \otimes_{K[Y]} A_m$

$$\simeq \underset{A_n}{\circlearrowleft} K[X] \boxtimes_K A_m$$

$$\simeq A_{n+m} \Big/ A_{n+m}(\partial_x)$$

Then $D_{Y\leftarrow X\times Y} \simeq A_{n+m} \Big/ (\partial_x) A_{n+m}$ so $\pi^* M \simeq K[X] \boxtimes M \quad (M \circlearrowleft A_m)$

$\pi_* N \simeq N \Big/ (\partial_x) N \quad (N \circlearrowleft A_{n+m})$

If M fin gen. (as A_m-mod) then

(1) $\pi^* M$ fin. gen. as A_{n+m}-mod

(2) $d(\pi^* M) = n + d(M)$ so holonomicity preserved

(3) $m(\pi^* M) \leq m(M)$.

However if N fin. gen, $\pi_* N$ is not fin. gen, if N holomorphic its true that $\pi_* N$ holonomic. (follows from "Fourier transform") of $\iota' : Y \hookrightarrow X\times Y$ for some $\iota'_* N'$.