Characteristic Varieties

Jiang 10. 24. 2012

① Filtrations

B, Γ, $\chi(t, \Gamma, M)$. $d(M)$.

"Gelfand - Kirillov dimension."

$$d(M) = \delta(M, V), \quad V = \text{span}\{x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, 1\}.$$

$$\delta(M, V) = \inf\{\ell \mid \dim_{\mathbb{C}} B_\ell \Gamma_0 \le k^\ell \quad \forall\, k >> 0\}.$$

It is a generalization of ~~Gelf~~ deg. of Hilbert poly.

(Vogan: Gelfand-Kirillov dim. of Harish-Chandra modules.
Invent. Math. 1978)

$\mathcal{U}(\mathfrak{g})$.

Thm: Bernstein Inequalities.

M: F.G. A_n-module. $(n>0)$.

Then $d(M) \geq n$.

When "$=$", call it holonomic.

$d(\mathbb{C}[x_1, \dots, x_n]) = n$.

Schrödinger Model?

$\widetilde{Sp_{2n}}(\mathbb{R})$ acting on $\mathbb{R}[x_1, \dots, x_n]$.

Gelfand Kirillov dim, minimal rep ...

Characteristic Varieties in a geometric interpretation of $d(M)$.

(A_n, B), (M, Γ).

$gr^{\Gamma}(M)$ is an S_n-module.

$$Ann(M,\Gamma) = \{ s \in S_n \mid s \cdot m = 0 \text{ for any } m \}$$

$rad(Ann(M,\Gamma))$ is independent of the choice of good filtrations.

denoted by $\underline{I(M)}$

<u>Define</u>. $Z(I(M)) = Ch(M)$. is the Affine Variety in $\mathbb{C}^{2n}$.

They talked about "Bible".

<u>Basic Properties</u>:

① $0 \to N \to M \to M/N \to 0$

$$Ch(M) = Ch(N) \cup Ch(M/N).$$

② (GTM 52, Chap 1. Thm 7.5).

$$\dim(Ch(M)) = d(M).$$

$\|$

Krull dim of $S_n/I(M)$

Geometric proof of Bernstein inequality. ④

(Not complete, eventually go to. Am. J. Math 1981

O. Gabler.)

involutive property. $T_p(Ch(M)) \supset T_p(Ch(M))^{\perp}$.

3. Construct Examples of non-holonomic A_n-modules.

① Stafford (1985) $n>2$, $\lambda_2, \cdots, \lambda_n$. alg. ind. $/\mathbb{Q}$.

$$\delta = \partial_1 + \left(\sum_{i=2}^{n} \lambda_i x_1 x_i \partial_i + x_i \right) + \sum_{i=2}^{n} (x_i - \partial_i)$$

$M = A_n / A_n \delta$. $d(M) = 2n-1$

J. Bernstein, V. Lunts. (1988)

More geometric Argument.

~~dim(M)~~ $d(M) = 2n-1$.

Should work for other dim between n & $2n-1$.

$V \subset \mathbb{C}^{2n}$ affine

Call it involutive, if $\forall$ non-singular p,

$T_p(V) \subset \mathbb{C}^{2n}$ is involutive

($\exists$. symplectic structure on $T_p^*(V)$

Prop 11.2.2. ~~dim(~~ V involutive aff in $\mathbb{C}^{2n}$

$$d(V) \geq 2n.$$

proof : $p \in V^{ns}$.

$$T_p(V)^{\perp} \subset T_p(V)$$

$$\Rightarrow \dim T_p(V) \geq n \Rightarrow \dim V \geq n.$$

An affine involutive variety of $\mathbb{C}^{2n}$, ~~is~~ called Laglangent if $\dim(V) = n$. then

$$T_p(V) = T_p(V)^{\perp} \text{ for all } p \in V^{ns}.$$

Prop 11.2.4.

An affine variety is involutive

$\Leftrightarrow$ ~~$I(V)$ is~~ $I(V)$ is stable under Poison bracket.

$\forall f, g \in S_n$, $\mathbb{C}^{2n} \underset{I}{\overset{\Phi}{\rightleftarrows}} (\mathbb{C}^{2n})^*$

$\{f, g\}(p) = \omega(I(d_p(f)), I(d_p(g)))$