A: ring $\quad J \subseteq A$ central, two-sided ideal. $\quad A/J = K$: field.

M: finite length A-mod. $\quad \mu \in \mathrm{End}_A(M)$, $\quad T_1, T_2 \in \mathrm{End}_{\mathbb{Z}}(M) \quad T_0 := [T_1, T_2]$

s.t. ① $\mu M = \ker \mu \qquad JM \subset \mu M$.

② $[T_i, \rho(A)] \subset \rho(J)$ and $[T_i, \rho(J)] = 0$

③ $[T_i, \mu] = 0 \qquad T_0(M) \subset \mu M$

④ T_1, T_2 nilpotent

$T_0 \in \mathrm{End}_A(M)$ inducing $\bar{T}_0 \in \mathrm{End}_K(\bar{M}) \qquad (\bar{M} := M/\mu M)$

Key lemma: One can choose $e_1, \dots, e_\ell \in M$ s.t. $\{\bar{e}_i\} \subset \bar{M}$ form K-basis and if we write $T_0 e_i = \mu \sum_j g_{ij} e_j$ with $g_{ij} \in A$ then

$\bar{T}_0 \bar{e}_i = \sum_j \pi(g_{ij}) e_j \qquad \pi : A \xrightarrow[\text{proj.}]{} K$

Then $G = (g_{ij}) \in \mathrm{Mat}_\ell(A)$ is of form $G = Z + F$, $\quad Z$: upper triangular
F: sum of commutators

(Immediately implies $\pi(\mathrm{tr}\, G) = 0$)

Sketch of pf.: If $T \in \mathrm{End}_{\mathbb{Z}}(M)$ commuting with μ then $T(\mu M) \subset \mu M$

so T induces endom. of $\mathrm{End}_{\mathbb{Z}}(\bar{M})$, call it $\tilde{T}$, such that $\tilde{T}\tau = \tau T$
with $\tau : M \to \bar{M}$.

(Warning: somewhat contrary to Gabber's notation!). This is satisfied by T_1, T_2 according to ③.

From ③ above, $\tilde{T}_1, \tilde{T}_2$ commute.

By ② + fact that $JM \subset \mu M$, then $\widetilde{\rho(A)}$ commutes with each $\tilde{T}_i$

so $\tilde{T}_i$ are K-linear.

Define subspaces of $\mathrm{End}_K(\bar{M})$ by:

$$V^0 := \widetilde{\rho(A)} = K$$

$$V^1 := K\cdot\tilde{T}_1 + K\cdot\tilde{T}_2$$

$$\vdots$$

$$V^n := (V^1)^n = \sum_{i+j=n} K\cdot\tilde{T}_1^{\,i}\,\tilde{T}_2^{\,j}$$

$T\begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} \in \bar{M}$, so expressible

as $\bar{C}\cdot\begin{pmatrix} \bar{e}_1 \\ \vdots \\ \bar{e}_\ell \end{pmatrix}$

for $\bar{C} \in \mathrm{Mat}_{\ell\times\ell}(K)$

take lift $C \in \mathrm{Mat}_{\ell\times\ell}(A)$. Then

$$\begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} \equiv C\begin{pmatrix} e_1 \\ \vdots \\ e_\ell \end{pmatrix} \mod \mu M.$$

But $\mu^2 = 0$ so

$$\begin{pmatrix} \mu x_1 \\ \vdots \\ \mu x_\ell \end{pmatrix} = \mu\cdot C\begin{pmatrix} e_1 \\ \vdots \\ e_\ell \end{pmatrix}$$

So we may rewrite:

$$\begin{pmatrix} T_1 e_1 \\ \vdots \\ T_1 e_\ell \end{pmatrix} = (E+\mu C)\begin{pmatrix} e_1 \\ \vdots \\ e_\ell \end{pmatrix}$$

Take commutator for T_1, T_2. Lemma follows.

$\tilde{T}_1, \tilde{T}_2$ nilpotent $\Rightarrow$ $V^n = 0$ for $n \gg 0$.

Get filtration of $\bar{M}$:

$$0 = V^n\bar{M} \subset V^{n-1}\bar{M} \subset \cdots \subset V^1\bar{M} \subset V_0\bar{M} = \bar{M}.$$

Choose basis according to successive quotients, give $\bar{e}_i$'s and then lift.

In particular, matrix of $\tilde{T}_1, \tilde{T}_2$ in this basis will be strictly upper triangular.

$(\tilde{T}_i V^j\bar{M} \subset V^{j+1}\bar{M}$ for $i=1,2)$

So eg.

$$\begin{pmatrix} T_1 e_1 \\ \vdots \\ T_1 e_\ell \end{pmatrix} = E\begin{pmatrix} e_1 \\ \vdots \\ e_\ell \end{pmatrix} + \begin{pmatrix} \mu x_1 \\ \vdots \\ \mu x_\ell \end{pmatrix} \quad (*)$$

$E \in \mathrm{Mat}_{\ell\times\ell}(A)$ $x_j \in M$

(strictly) upper triangular

same for T_2.
