$K := \bar{A} / \underline{m}$: max. ideal (with notations as in Proposition 3)

$\pi : A \to \bar{A}$ can. projection.

$q : \bar{A} \to K$

$0 < s :=$ length ~~of~~ of $\bar{M} := M / \mu M$ μ: central

For any $a \in A$, $\rho(a) : M \to M$ $\rho(a)(x) = ax$, $\rho(a) \in \mathrm{End}_{\mathbb{Z}}(M)$

and $\in \mathrm{End}_A(M)$ if a is $\underline{\text{central}}$. In particular $\rho(\mu) \in \mathrm{End}_A(M)$,

and $\ker \rho(\mu) = \mathrm{Im}\ \rho(\mu)$.

($\underline{\text{Rmk}}$: $0 \to \mu M \to M \to \bar{M} \to 0$ + $\bar{M} \simeq \mu M \Rightarrow$ ($\bar{M}$ finite length $\Leftrightarrow$ M is finite length))

Set $J := \mu A$. $[J, A] = [\mu A, A] = \mu [A, A] \subseteq \mu^2 A = 0$

so $J \subseteq Z(A)$: center

for every $a, b \in A$, $[a,b] \in J$, hence $\rho([a,b]) \in \mathrm{End}_A(M)$.

Put $[a,b] = \mu c$ so that $\{\bar{a}, \bar{b}\}_\mu = \bar{c}$.

$\rho([a,b])(M) \subset \rho(\mu)(M) = \mu M$. So we have $\bar{\rho}([a,b]) \in \mathrm{End}_{\bar{A}} \bar{M}$ which is just mult. by $\bar{c}$.

Now $\underline{m}^s \bar{M} = 0$. For each $i = 0, \dots, s-1$, we have the K-vector space $\underbrace{\underline{m}^i \bar{M} / \underline{m}^{i+1} \bar{M}}_{W_i}$ with

$$\sum_{i=0}^{s-1} \dim_K \underline{m}^i \bar{M} / \underline{m}^{i+1} \bar{M} = s.$$

Let $\bar{\rho}([a,b])_i$ be the induced K-linear map on W_i, $i = 0, \dots, s-1$ which is just mult. by $q(\bar{c}) \in K$.

$$tr'(\bar{\rho}([a,b])) := \sum_i tr_K(\bar{\rho}([a,b])_i) = s\, q(\bar{c}).$$

So $tr' = 0$ only if $q(\bar{c}) = 0$

We need $q(\bar{c}) = 0$ if $\bar{a}, \bar{b} \in \underline{m}$. Hence we are finished if we show:

if $a, b \in \pi^{-1}(\underline{m}) \subset A$, then $tr'(\rho([a,b])) = 0$.

Key point: if $a \in \pi^{-1}(\underline{m})$, then $\rho(a) \in \mathrm{End}_{\mathbb{Z}}(M)$ is nilpotent.

since $\underline{m}^s \overline{M} = 0$, and so $\pi^{-1}(\underline{m})^s M \subset \mu M$. Hence $\pi^{-1}(\underline{m})^{2s} M \subseteq \rho(\mu)^2 M = 0$.

(hence "uniformly" nilpotent)

$2s$ works for all elts of $\pi^{-1}(\underline{m})$

We rephrase the problem: A: ring. $J \subset A$ two-sided ideal, $J \subset Z(A)$, and such that A/J is comm., local ring, $\underline{m}$: maximal ideal. $K = A/\underline{m}$ with $\mathrm{char}(K) = 0$.

M: finite length A-module.

Two endomorphisms $T_1 = \rho(a)$, $T_2 = \rho(b) \in \mathrm{End}_{\mathbb{Z}}(M)$.

$\mu = \rho(\mu) \in \mathrm{End}_A(M)$ satisfying:

1. $\mathrm{Im}\ \mu = \mathrm{Ker}\ \mu$ and $JM \subseteq \mathrm{Im}\ \mu = \mu M$
2. $[T_i, \rho(A)] \subseteq \rho(J)$ and $[T_i, \rho(J)] = 0$ for $i = 1, 2$

   ($\rho(A)$: mult. by elts in A, as elts of $\mathrm{End}_{\mathbb{Z}}(M)$)
3. $[T_i, \mu] = 0$ $(i = 1, 2)$ and $[T_1, T_2](M) \subseteq \mu M$. (Let $T_0 := [T_1, T_2]$ which is A-linear) and induces $\overline{T}_0 \in \mathrm{End}_{\bar{A}}(\overline{M})$ where $\bar{A} := A/J$, $\overline{M} := M/\mu M$.
4. T_1, T_2 nilpotent.

Want: $tr'(\overline{T}_0)$ (defined as before via successive quotients) $= 0$.

By replacing A with $A/\pi^{-1}(\underline{m})^{2s}$ and J by $J + \pi^{-1}(\underline{m})^{2s} / \pi^{-1}(\underline{m})^{2s}$

we may assume $\underline{m}$ is nilpotent. $\Rightarrow$ $\bar{A}$: complete local ring of equal char.

$\Rightarrow$ $\exists$ subfield $F \subset A/J$ complementary to $\underline{m}$ where

$q : A/J \longrightarrow (A/J)/_{\underline{m}} = K$ induces an isom. $q : F \xrightarrow{\sim} K$

and thus $tr'(\bar{T}_0) = q(tr_F(\bar{T}_0))$

Let $B := \pi^{-1}(F) \subset A$ (subring). $J \subset B$ central two-sided ideal.

and $B/J = F$ is a field.

M has finite length as B-module.

All data μ, T_1, T_2 satisfies same properties ①–④ above relative to B.

So replacing A by B in all instances above, we may assume $A/J = K$ is a field.

(i.e. regard $\bar{T}_0 \in \mathrm{End}_K(\bar{M})$. So can use linear algebra / choice of basis to compute trace)

For $x \in M$ let $\tau : M \longrightarrow \bar{M} = M/\mu M$. We assumed $T_0(M) \subset \mu M$ so

$x \longmapsto \tau(x) = \bar{x}$ can write $T_0(x) = \mu(y)$ for some $y \in M$.

Thus $\bar{T}_0(\bar{x}) = \bar{y}$.

Let $\{e_1, \dots, e_\ell\} \subset M$ be s.t. $\{\bar{e}_1, \dots, \bar{e}_\ell\}$ is a K-basis of $\bar{M}$.

$T_0 e_i = \mu \sum_j g_{ij} e_j$ $g_{ij} \in A$, so that $\bar{T}_0 \bar{e}_i = \sum_j \pi(g_{ij}) \bar{e}_j$

then $tr(\bar{T}_0) = \pi(tr\, G)$ $G = (g_{ij})$

Key Lemma: One can choose $e_1, \dots, e_\ell$ s.t. $G = Z + F$ with Z: upper triangular

F: sum of two commutators of matrices.