Prop 1: Let B be a k-alg. with central non-zero divisor ν s.t. $\overline{B} := B/\nu B$ is commutative, Noetherian. Let P be a fin-gen B-module on which ν acts injectively and put $\overline{P} = P/\nu P$. Then $\sqrt{\mathrm{Ann}_{\overline{B}}(\overline{P})}$ is closed under $\{\ ,\ \}_\nu$. (the first reduction...)

Second reduction: Let B, ν, P as in Proposition 1. Let $R := B/\nu^2 B$ with $\pi : B \to R$ canonical proj. $\mu := \pi(\nu) \in R$. Then μ central with $\mu^2 = 0$.

Let $\mathrm{Ann}_R(\mu) := \{ r \in R \text{ s.t. } \mu r = 0 \}$. Then $\mu R \subseteq \mathrm{Ann}_R(\mu)$.

(this is true for any R, μ with $\mu^2 = 0$)

claim: For our choice of R, μ, the reverse inclusion is true as well.

(pf: write $r = \pi(b)$, $b \in B$ $\quad 0 = \mu r = \pi(\nu b) \Rightarrow \nu b \in \nu^2 B$

i.e. $\nu b = \nu^2 b_1$, and since ν is non-zero divisor $b = \nu b_1 \Rightarrow r = \mu\, \pi(b_1) \in \mu R$)

$\overline{R} := R/\mu R \underset{\phi}{\xrightarrow{\sim}} B/\nu B$ so $\overline{R}$ is comm., Noetherian k-algebra.

Given $x, y \in \overline{R}$, choose lifts $r, q \in R$ (write $\overline{r} = x$, $\overline{q} = y$). Then

$[r, q] = rq - qr = \mu s$ for some $s \in R$ determined up to $\mathrm{Ann}_R(\mu) = \mu R$.

$\Rightarrow \overline{\mu^{-1}[r,q]} =: \overline{s} \in \overline{R}$ is well defined.

Put $\{x, y\}_\mu := \overline{\mu^{-1}[r,q]}$. Under isomorphism ϕ, the two brackets $\{\ ,\ \}_\mu$ and $\{\ ,\ \}_\nu$ correspond.

Let $N := P/v^2P$, N is fin. gen., Noetherian module for R and

$\bar{N} := N/\mu N$ is fin-gen $\bar{R}$-mod with $\bar{N} \simeq P/vP =: \bar{P}$ under the

isomorphism $\bar{R} \xrightarrow{\sim} \bar{B}$. By similar argument to before,

if $\mathrm{Ann}_N(\mu) := \{ n \in N \mid un = 0 \}$ then $\mathrm{Ann}_N(\cdot\mu) = \mu N$ (using again that v acts injectively on P)

<u>Prop. 2</u>: (which implies Prop. 1' by above discussion)

Let R be a k-alg. with central elt μ s.t. $\mu^2 = 0$ and s.t.

$\bar{R} = R/\mu R$ is comm., Noetherian. Let N be a fin. gen. R-mod.

<u>Assume</u>: (1) $\mathrm{Ann}_R(\mu) = \mu R$ (ensuring Poisson bracket)

(2) $\mathrm{Ann}_N(\mu) = \mu N$

Then $\sqrt{\mathrm{Ann}_{\bar{R}}(\bar{N})}$ is closed under $\{\ ,\ \}_\mu$ (essentially thm. 2 in Gabber.)

<u>Remark</u>: Gabber only assumes (2). Hence $\{\ ,\ \}_\mu$ may only be assumed to take values in $\bar{R}/I$ with $I = \mathrm{Ann}_R(\mu)/\mu R$. (And also weaker assumptions on Noetherian.)

Third <u>reduction</u> (localization)

Since $\sqrt{\mathrm{Ann}_{\bar{R}}(\bar{N})} = \bigcap\limits_{\substack{\mathfrak{p}:\text{min.} \\ \text{prime of } \bar{N}}} \mathfrak{p}$, it suffices to check that $\{\mathfrak{p}, \mathfrak{p}\}_\mu \subset \mathfrak{p}$ for all such minimal primes $\mathfrak{p}$.

Fix one such $\mathfrak{p}$.

Rename: $\pi : R \longrightarrow \bar{R}$: canonical proj. $S := \pi^{-1}(\bar{R} \setminus \mathfrak{p})$, a mult. closed subset in R

form local ring $\bar{R}_\mathfrak{p} = \pi(S)^{-1}\bar{R}$ and <u>also</u> have ring of fractions $S^{-1}R$ (even though R not nec. commutative!) ($1 \in S$, $st \in S$ if $s, t \in S$.)

Construction is something like that of derived category.

Works in our case because for our particular R, any multiplicatively closed set S satisfies the Ore conditions:

(i) $\forall\ s \in S,\ a \in R \quad Sa \cap Rs \neq \phi$. i.e. $\exists\ t \in S,\ b \in R$ s.t. $ta = bs$

(ii) if $a \in R$, $s \in S$ with $as = 0$ then $\exists\ t \in S$ with $ta = 0$.

These are sufficient to define the localization $S^{-1}R$.

These two conditions are satisfied by our ~~R~~ R since, $\forall\ a \in R$

$ad(a) : R \to R$ given by $ad(a)(x) = ax - xa$ is nilpotent : $ad(a)^2 = 0$.

(recall that μ central so $\overline{ad(a)} = 0$ in $\bar{R}$ gives result.)

Commutative

hence we may solve equations in Ore conditions.

$S^{-1}R$ comes with morphism $\varphi : R \to S^{-1}R$

s.t. (1) $\varphi(s)$ invertible $\forall\ s \in S$

(2) every elt. of $S^{-1}R$ can be written $\varphi(s)^{-1}\varphi(a)$ for some $a \in R, s \in S$

(3) $\varphi(a) = 0 \iff sa = 0$ for some $s \in S$.

+ univ. prop. : $\forall$ rings C and $\psi : R \to C$ s.t. $\psi(s)$ invertible $\forall\ s \in S$

$\exists!\ \sigma : S^{-1}R \to C$ s.t. $\sigma \circ \varphi = \psi$.

If N is a left R-module, put $S^{-1}N := S^{-1}R \otimes_R N$. Then since $S^{-1}R$ is flat as right R-module, if $0 \to N' \to N \to N'' \to 0$ is exact, so is

$0 \to S^{-1}N' \to S^{-1}N \to S^{-1}N'' \to 0$.

Let $s \in S$. $\mu s = s\mu \Rightarrow \varphi(\mu)\varphi(s) = \varphi(s)\varphi(\mu) \Rightarrow \varphi(s)^{-1}\varphi(\mu) = \varphi(\mu)\varphi(s)^{-1}$

$\forall\ a \in R \quad \varphi(\mu)\left(\varphi^{-1}(s)\varphi(a)\right) = \varphi(s)^{-1}\left(\varphi(\mu a)\right) = \varphi(s)^{-1}\varphi(a\mu) = \left(\varphi(s)^{-1}\varphi(a)\right)\varphi(\mu)$

i.e. $\varphi(\mu)$ central in $S^{-1}R$.

Let $\overline{S^{-1}R} := S^{-1}R \big/ \varphi(\mu) S^{-1}R$. Then we have:

$$\begin{tikzcd}
R \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \bar{R} \arrow[d, "\bar{\varphi}"] \\
S^{-1}R \arrow[r, "\sigma"] \arrow[d] & \bar{R}_{\mathfrak{p}} \\
\overline{S^{-1}R} \arrow[ur, "\xi", "\wr"'] &
\end{tikzcd}$$

$\underset{\sim\sim}{\overline{S^{-1}R}}$ commutative, Noetherian, local

canonical isom ξ is guaranteed by universal property.

By flatness of $S^{-1}R$,

$$\mathrm{Ann}_{S^{-1}R}(\varphi(\mu)) = S^{-1}R \cdot \varphi(\mathrm{Ann}_R(\mu))$$

$$= S^{-1}R \cdot \varphi(\mu R) = \varphi(\mu) S^{-1}R$$

Similarly, $\mathrm{Ann}_{S^{-1}N}(\varphi(\mu)) = \varphi(\mu) S^{-1}N$

so we have Poisson brackets

$\{\ ,\ \}_{\varphi(\mu)}$ and $\{\ ,\ \}_{\mu}$ correspond under the isomorphism ξ.

__Proposition 3__: Let A be a k-alg. with central elt μ, $\mu^2 = 0$ and $\bar{A} = A/\mu A$ is comm., Noetherian, local k-alg. with max. ideal $\mathfrak{M}$. Suppose $\mathrm{Ann}_A(\mu) = \mu A$ and $\exists$ fin.-gen. module M s.t. $\mathrm{Ann}_M(\mu) = \mu M$ and $\bar{M} := M/\mu M$ is non-zero. (and nec. of finite length)

Then $\mathfrak{M}$, the maximal ideal, is closed under poisson bracket.

(which implies Proposition 2 by above discussion)