1. Poisson algebras

Let B be comm. alg. $/ k$: char. 0 with unit

Assume we also have Lie bracket $\{ , \}$ on B. Further require that

$\{ - , b \}$ is a derivation.

i.e. $\{ a_1 a_2, b \} = a_1 \{ a_2, b \} + a_2 \{ a_1, b \}$.

Then $(B, \cdot, \{ , \})$ is Poisson algebra.

Example: A : filtered assoc. k-alg. (integer filtration)

$A = \bigcup_{i \in \mathbb{Z}} A_i$ A_i : additive sbgps with $A_i \subset A_{i+1}$
$A_i A_j \subset A_{i+j}$ $\forall i, j$.
$k \subset A_0$.

with assoc. graded $gr A = \bigoplus_{i \in \mathbb{Z}} A_i / A_{i-1}$
k-alg.

Assumption: $gr A$ is commutative. (i.e. A is "almost commutative")

Define bracket : $f \in A_i / A_{i-1}$, $g \in A_j / A_{j-1}$ choose lifts a, b resp. for f, g.

$ab - ba \in A_{i+j-1}$ by commutativity. $\{ f, g \} = \sigma(ab - ba)$
$\in A_{i+j-1} / A_{i+j-2}$

(easy to check well-defined — i.e. indep. of lift)

hence $gr A$ is Poisson algebra. ← This will be our main example.

2. Statement of Gabber's Thm :

M a fin. gen. A-mod with good filtration Γ. $M = \bigcup_{i \in \mathbb{Z}} \Gamma_i(M)$,
$\Gamma_i(M) \subseteq \Gamma_{i+1}(M)$ $\forall i$

"Good" means $\Gamma_i(M) = \sum_{j=1}^{r} A_{i - n_j} \cdot m_j$ $\forall i$

where $n_j \in \mathbb{Z}$, $\{ m_1, \dots, m_r \}$: fixed set of generators of M.

$\mathcal{J}(M) := \sqrt{\mathrm{Ann}(\mathrm{gr}^{\Gamma} M)}$ is a graded ideal in $\mathrm{gr}\, A$, indep. of good filtration.

<u>Thm. (Gabber)</u>: If $\mathrm{gr}(A)$ is Noetherian (and commutative) then $\{ \mathcal{J}(M), \mathcal{J}(M) \} \subset \mathcal{J}(M)$.

(cf. Coutinho for example in which an ideal does not have such property for radical. $J = (y_1^2, y_2^2, y_1 y_2)$ but $y_1, y_2 \in \sqrt{J}$ with $\{y_1, y_2\} = 1 \notin \sqrt{J}$)

3. First reduction:

Definition: The Rees ring of A is the graded ring $B = \bigoplus_{i \in \mathbb{Z}} A_i$ (with mult. induced from mult. on A)

$\lambda_i : A_i \hookrightarrow B$: canonical injections

$1_B = \lambda_0(1)$ Set $v := \lambda_1(1)$. (central, non-zero divisor in B) (mult. by v is injective)

Claim (easy):

① $\exists$ canonical isom. of graded rings $\bar{B} := B/vB \xrightarrow{\sim} \mathrm{gr}\, A$ so $\bar{B}$, under our assumptions, is comm., noeth.

② Let M be an A-module w/ filtration Γ

Put $\Gamma M := \bigoplus_{i \in \mathbb{Z}} \Gamma_i M$ with $\lambda_i : \Gamma_i M \hookrightarrow \Gamma M$

$\lambda_i(a) \lambda_j(m) := \lambda_{i+j}(am)$ makes ΓM into a B-module.

The action of v on ΓM is injective, and have canonical isom. of graded mods:

$\overline{\Gamma M} := \Gamma M / v \Gamma M \xrightarrow{\sim} \mathrm{gr}^{\Gamma} M$ (after identification of $\bar{B}$, $\mathrm{gr}\, A$ via above isom.)

③ M is finitely gen, Γ good $\Longleftrightarrow$ ΓM is fin. gen B-mod.

In this case, $\overline{\Gamma M}$ is fin. gen. over $\bar{B}$, hence Noetherian.

Finally, under the isomorphism, $\mathcal{J}(M) = \sqrt{\mathrm{Ann}_{\overline{B}}(\overline{\Gamma M})}$.

(these steps have removed need for talking about grading)

B: any ring w/ central non-zero divisor v

$\phi_v : B \longmapsto vB$ is bijection. For $x \in vB$, put $v^{-1}x := \phi_v^{-1}(x)$

$a \longmapsto va$

Assume $\overline{B} = B/_{vB}$ is commutative. Then $\underset{\substack{\| \\ ab-ba}}{[a,b]} \in vB \quad \forall\, a,b \in B$.

So we may define Lie bracket on $\overline{B}$
(depending on v) by $\{a,b\}_v := \overline{v^{-1}[a,b]}$, giving Poisson structure.

For B and $v := \lambda_1(1)$ as before, this Poisson structure matches that of $\mathrm{gr}\, A$ with Poisson bracket under isomorphism.

Prop. 1: Let B be a k-alg. with a central, non-zero divisor v s.t. $\overline{B} := B/_{vB}$ is comm., Noetherian k-alg. Let P be a fin gen. B module on which v acts injectively. Set $\overline{P} = P/_{vP}$. Then $\sqrt{\mathrm{Ann}_{\overline{B}}(\overline{P})}$ is closed under $\{\ ,\ \}_v$.