Holonomic modules

Def) A left A_n-module of degree n is called Holonomic. Zero module is holonomic.

$\boxed{n \text{ is minimal by Bernstein}}$

Easy example) $K[x_1, \dots, x_n]$ is a holonomic module.

$A_n(K) \overset{\text{Ring}}{\subset} \mathrm{End}_K K[X]$

Note: A_n not holonomic

For $n=1$, many examples: ~~Fo~~

Let $I \neq 0$ be a left ideal of A_1

Recall that $d(A_1/I) \leq 2 \cdot 1 - 1 = 1$

$\Rightarrow d(A_1/I) = 1$. $\Rightarrow A_1/I$ holonomic

Prop (1) Submodules and quotients of holonomic A_n-modules are holonomic. $\boxed{\text{Exact seq.}}$

(2) $\sum$ (~~Holonomic~~)$_i$ is holonomic

$\oplus$ Holonomic

Pf (1) Let M holonomic A_n-module, $N < M$ submodule. By 9.3.2,

$$n = d(M) = \max\{d(N), d(M/N)\} \geq d(N) = n$$
$$\geq d(M/N) = n$$

By Bernstein N, M/N holonomic

(2) Recall by 9.3.3,

$$d(M_1 \oplus \cdots \oplus M_k) = \max_i d(M_i),$$

and $\sum$ Holonomic : $\oplus$ Holonomic

Corollary Finitely generated torsion A_1-modules are holonomic ③

Pf Let $\{u_i\}_1^n$ be generators of M f.g. ~~tors~~ tor/A_1.

Let $\{b_i\}_1^n$ be associated annihilators, i.e.

$$b_i u_i = 0$$

~~A~~ Thus ~~$A_1 u_i$~~ is a quotient of

$$A_i u_i \cong \underbrace{(A_1/A_1 b_i)}_{\text{holonomic}} \Big/ (\ast), \quad \text{Thus}$$

$A_1 u_i$ is holonomic & so is the sum.

$\blacksquare$

~~Thus for f.g. A_1 modules~~

~~holonomic $\Leftrightarrow$ torsion~~

~~Observe that if $I \subseteq J$ are left ideals of A_1, then I/J is torsion~~

J/I torsion holonomic

Prop Holonomic A_n-modules are torsion

Proof Let M holonomic $/A_n$. Let $0 \neq u \in M$.

$$\phi : A_n \to M,$$
$$a \mapsto au,$$

Clearly $\phi(A_n) \subseteq M$, so $d(\phi(A_n)) = n$.

Since $d(A_n) \overset{9.3.2}{=} \max\{ d(A_n/\ker\phi), d(\ker\phi)\}$

$\Rightarrow d(\ker\phi) = 2n$.

Thus $\ker\phi$ is non-trivial, so u is torsion.

For ~~[illegible]~~ M f.g over A_1

torsion ~ holonomic

Not true for $n \geq 2$

E.g.) $M = A_n / (A_n \partial_n)$

Pf) Ex (9.5.2-3)

Recall A module M over ring R is <u>Artinian</u> if all descending ~~sequ~~ chains of submodules

$$N_1 \supseteq N_2 \supseteq \cdots$$

is stationary

Thm Let M left-module over ring R, N submod. of M.

(1) M Artinian $\Longleftrightarrow$ every set of submodules has ~~an minimal element element not containing~~ a minimal element.

(2) M Artinian $\Longleftrightarrow$ N and M/N Artinian

(3) For N' a submodule s.t. $M = N + N'$;

N', N Artinian $\Longrightarrow$ M Artinian.

Pf (1) "$\Rightarrow$" M Artinian, $S \neq \emptyset$ collection of submodules. ⑦

Suppose there is no minimal element.

Choose a submodule $N_n \in S$. Not minimal, so $\exists$ submodule $N_{n+1} \in S$ s.t. $N_{n+1} \subsetneq N_n$. $\Rightarrow \Leftarrow$

"$\Leftarrow$"

Let

$N_1 \supseteq N_2 \supseteq \cdots$ be chain of submodules.

Let $S = \{N_i\}_{i=1}^{\infty}$. Let N_n be minimal guy.

obv. stationary.

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(2) "$\Rightarrow$" M Artinian, N submodule.

~~Any chain of submodules of N is a chain of submodules of M, so N Artinian.~~

~~Let π:~~ $0 \to N \xrightarrow{f} M \xrightarrow{g} N'' \to 0$

M Artinian $\Rightarrow$ N'' Artinian.

Let $N_0'' \supseteq N_1'' \supseteq \cdots$ be descending chain.

Let $M_i = g^{-1}(N_i'')$. So $M_i \supsetneq M_{i+1}$ $\Rightarrow \Leftarrow$

Conversely, let ~~...~~ N', N'' be Artinian.

Let ~~...~~ $M_1 \supseteq M_2 \supseteq \cdots$ of ~~...~~ M.

Let ~~...~~ $N_i'' = g(M_i)$. This is stationary,

so $M_i + N' = M_{i+1} + N'$ for $i \gg 0$.

N' Artinian $\Rightarrow$ $f^{-1}(M_i) = f^{-1}(M_{i+1})$ for $i \gg 0$,

so M_i must terminate

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(7) ~~...~~ Let $M = N + N'$, N, N' Art.

Then $M/N \cong \left(N'/N' \cap N\right)$ Artinian,

So M is.

$\boxed{\text{Thm}}$ Holonomic Modules are Artinian

$\boxed{\text{Pf}}$ Let ~~M~~ $\overset{A_n\text{-module}}{M} \supseteq \underbrace{N_0 \supseteq N_1 \supseteq \cdots \supseteq N_r}_{\text{all have } d(\cdot)=n}$

So $m(N_i) = m(N_{i+1}) + m(N_i/N_{i+1})$

Thus $m(M) = \sum_{0}^{r-1} m(N_i/N_{i+1}) + m(N_r) \geq r$

Not all A_n-modules are Artinian

E.g. A_n over itself

$$A_n x_n \supseteq A_n x_n^2 \supseteq A_n x_n^3 \supseteq \cdots$$

~~Definition length~~

Consider ~~that~~ $N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k$

Length

M left module over R
- Artinian
- Noetherian

$N_i < M$

(N_{i+1}/N_i) is simple

Consider

$S = \{\text{submodules of } \underbrace{M/N_k}_{\text{Artinian}}\}$,

Let N_{k+1} be minimal element. Then we obtain

$N_0 \subseteq \cdots \subseteq N_k \subseteq N_{k+1}$.

Noetherian $\Rightarrow$ terminates,

$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = M$

~~Fact~~

called composition series

The # of gaps (r) is called the length of M

$\boxed{\underline{\text{Fact}}\ \text{(Artin–Schreier) Well-defined}}$

Scholium $L(M) \leq m(M)$ for M holonomic.

Pf

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$$

Have: ① M_i holonomic
② M_i/M_{i-1} holonomic

$$\Rightarrow m(M) = \sum_{1}^{\ell} m(M_i/M_{i-1}) \geq \ell$$

Cor Let M Holonomic A_n-module M, $m(M) = 1$
$\Rightarrow M$ simple

Pf Suppose $0 \neq N < M$, Then $N, M/N$ holonomic,

so $\underset{1}{\underset{\|}{m(M)}} = \underset{0}{\underset{\|}{m(M/N)}} + \underset{1}{\underset{\|}{m(N)}} \Rightarrow M/N = 0.$

$\Rightarrow M = N.$

So M is simple

Thm Let R be simple left Noetherian and M f.g. R-module. Then

(as a left R module)

$$\left(M \text{ Artinian}, R \text{ not } \cancel{R}\text{Artinian}\right) \Longrightarrow M \text{ cyclic}$$

Pf Observe that $\ell(M)$ is well defined.

Idea: If $M = Ru + Rv$, want to show that $\exists a \in R$ s.t.

$M = R(u + av)$.

Induct on length: If $\ell(M) = 1 \Rightarrow M$ irr $\Rightarrow M = Ra$

for $\underset{M}{a \neq 0}$ ✓

Suppose the statement holds for length $< r$, and

$M = Ru + Rv$, $\ell(M) = R$.

$\Rightarrow \ell(Rv)$ finite $\Rightarrow \exists c \in R$ s.t. $Rcv \overset{irr}{<} Rv$.

Thus $\ell(M/Rcv) < r$, so (by ind. hypothesis)

$$M = R(u + \lambda v) + Rv$$

for some $\lambda \in R$.

Rewriting, $M = Ru + \underbrace{Rv}_{irr.}$

Let $\underset{\text{morphism}}{\phi}: R \to M$

$r \mapsto ru$

Note: ϕ cannot be injective, since $(\text{im}\,\phi) \overset{\text{submodule,}}{<} M$ must be Artinian.

Thus $\exists\ 0 \neq d \in \operatorname{Ker} \phi$.

~~Claim: $R(u+bv) = M$~~

R simple $\Rightarrow RdR = R \Rightarrow RdRv \neq 0$.

Thus $\exists\, b \in R$ s.t. $dbv \neq 0$.

Claim: $M = R(u+dv)$.

$$d(u+bv) = \underset{\underset{Rv}{\cap}}{dbv} \neq 0.$$

Thus $Rdbv = Rv \Rightarrow v \in R(u+bv)$

$\Rightarrow u \in R(u+bv)$

$\blacksquare$

Cor: Holonomic modules are cyclic

Pf: M holonomic must be Artinian, A_n not Artinian ✓

Examples

Lemma Let M be a left A_n-module with a filtration Γ wrt. Bernstein filtration of A_n. Suppose $\exists c_1, c_2$ constants s.t.

$$\dim_K \Gamma_j \leq \frac{c_1 j^n}{n!} + c_2 (j+1)^{n-1} \qquad \left(\text{for } j >> 0\right)$$

Then M is holonomic ~~A~~ of multiplicity $\leq c_1$.

~~Cor: M is f.g.~~

Proof Claim 1: Every f.g. submodule of M ~~is holonomic~~ satisfies theorem.

Proof of claim:

Let $N < M$ be f.g. Then N admits a good filtration (Chapter 8) Ω_j w/ Hilbert poly. $\chi(t)$. By 8.32, $\exists r \in \mathbb{Z}^+$ s.t. $\Omega_j \subseteq \Gamma_{j+r} \cap N$.

$$\Rightarrow \boxed{\dim_K \Omega_j \leq \dim_K \Gamma_{j+r}}$$

$$\chi(j) = \dim_K \Omega_j \leq \cancel{\dim_K \Gamma_{j+1}}$$

$$\frac{c_1 (j+r)^n}{n!} + c_2 (j+r)^{n-1} \qquad j >> 0$$

$\Rightarrow \deg \chi(t) \leq n$.

Bernstein $\Rightarrow d(N) = n = \deg \chi(t) \Rightarrow m(N) \leq c_1$ ✓

Now consider ascending chain

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r \quad \text{f.g. submodules.}$$

holonomic deg n,

$$\Rightarrow m(N_i) = m(N_{i-1}) + m(N_i/N_{i-1})$$

$$\Rightarrow \sum_{2}^{r} m(N_i/N_{i-1}) + m(N_1) = m(N_r) \leq c_1$$

$\Rightarrow$ all ascending chains of f.g. submodules have length $< c_1$

~~$\Rightarrow M$ is f.g.~~

$\Rightarrow M$ is f.g.

Let $k(X) = k(x_1, \ldots, x_n)$ be the field of rat'l fns. The left action of A_n on $k[X]$ extends naturally, with x_i acting by mult, and $\partial_i \cdot \frac{f}{g}$ by quotient rule

$$\partial_i\left(\frac{f}{g}\right) = \frac{\partial_i(f)g - f\partial_i(g)}{g^2}$$

Fixing some nonzero polynomial $p \in k[X]$, one can consider

$$k[X, p^{-1}] = \{ f/p^r \mid f \in k[X] \}$$

Since partial derivatives of such fn's have denominator p^{2r} $\Rightarrow$ $k[X, p^{-1}]$ is a left A_n-submodule of $k(X)$.

Thm $M = K[X, p^{-1}]$ is holonomic w/ $m(M) \leq (\deg(p)+1)^n$

Proof Let $m = \deg p$,

let $\Gamma_k = \{ f/p^k \mid \deg(f) \leq (m+1)K \}$

Claim: Γ is a filtration of M,

① Let $f/p^k \in M$, $(\deg f) = s$. Then

$$f/p^k = \frac{f \cdot p^s}{p^{s+k}},$$

Now $\deg(fp^s) = sm + s = s(m+1) \leq (m+1)(s+k)$

$\Longrightarrow f/p^k \in \Gamma_{s+k}$. $\Longrightarrow M = \bigcup_{k \geq 0} \Gamma_k$. ✓

② Let $f/p^k \in \Gamma_k$, $\Longleftrightarrow \deg(f) \leq (m+1)k$

$$\Longrightarrow x_i(f/p^k) = \frac{x_i f p}{p^{k+1}} \in \Gamma_{k+1}$$

and $$\partial_i\left(\frac{f}{p^k}\right) = \frac{p^k \partial_i(f) - f\, \partial_i(p^k)}{p^{2k}}$$

$$= \frac{p\, \partial_i(f) - fk\, \partial_i(p)}{p^{k+1}}$$

where numerator has $\deg \leq (m+1)k + (m-1)$

$$\leq (m+1)(k+1)$$

$\Rightarrow \partial_i (f/p^k) \in \Gamma_{k+1}$.

Thus $B_1 \cdot \Gamma_k \subseteq \Gamma_{k+1} \Rightarrow B_i \Gamma_k \subseteq \Gamma_{i+k}$ ✓

(3) $\dim(\Gamma_k) \leq \dim$ vect. space of poly. of $\deg \leq (m+1)k$

$\Rightarrow \Gamma_k$ is f.d.

So Γ is a filtration and

$$\dim_k \Gamma_k \leq \binom{(m+1)k+n}{n}$$

Calculation shows:

$$\dim_k \Gamma_k \leq \frac{(m+1)^n k^n}{n!} + \frac{(m+1)^{n-1}(n+1)n(k+1)^{n-1}}{n!}$$

for $k \gg 0$

$\Rightarrow$ (by Lemma 3.1) that

M is holonomic, $m(M) \leq (m+1)$. ∎

Bernstein polynomial

Fix $p \in k[X]$. Let s be a variable, $K(s)$ field of rat'l fns.

Let $A_n(s)$ be the Weyl algebra over $K(s)$.

Let $A_n(K(s))p^s$ denote the $A_n(K(s))$-module generated by p^s, a formal symbol, s.t.

$$\partial_j \cdot p^s = s p^{-1} \partial_j(p) \cdot p^s .$$

~~Let t be an automorphism of~~

$$A_n(K(s))p^s < K(s)[X, p^{-1}]p^s ,$$

Let t be the aut. of $K(s)[X, p^{-1}]p^s$ s.t.

$$t(s^i p^s) = (s+1)^i p \cdot p^s$$

N.B. $A_n(k)$-linear, <u>not</u> $A_n(K(s))$-lin.

Thm ~~Let~~ $\cancel{p \in K[X]}$.

There exist $B(s) \in K[s]$ and a differential operator

polyn.

$D(s)$ in $A_n(K)[s]$ s.t.

$$B(s)p^s = D(s)pp^s$$

Proof Fact: $K(s)[X, p^{-1}]p^s$ is holonomic. Analogous to proof that $K[X, p^{-1}]$ is holonomic.

$\Rightarrow A_n(s)p^s$ is holonomic. Thus it has finite length, and

$$A_n(s)p^s \supseteq A_n(s)p \cdot p^s \supseteq A_n(s)p^2 \cdot p^s \supseteq \cdots$$

is stationary.

So $\exists K > 0$ s.t.

$$p^K \cdot p^s \in A_n(s)p^{K+1} \cdot p^s$$

$\Rightarrow$ (after applying t^{-K}) that $p^s \in A_n(s)p \cdot p^s$.

We can clear denominators and obtain that $\exists\ B(s) \in K[s]$ s.t.

$$B(s)p^s \in A_n(K)[s]p \cdot p^s.$$

$\blacksquare$

Note: $(B(s), D(s))$ not uniquely determined, but set of $B(s)$ is an ideal and has a unique monic generator, called the Bernstein polynomial.

<u>Hard to calculate generally.</u>