Starting point: M non-zero A_n-module with filtration $\{M_i\}$ of M w.r.t. Bernstein filtration, and positive integers e, c s.t.

$$\dim_k(M_r) \leq e \cdot \binom{r}{n} + c \cdot \sum_{i=0}^{n-1} \binom{r}{i} \quad \forall r.$$

Then M has finite length bounded by e.

(proven in John Goes' lecture.) $\leftarrow$ corollary to Bernstein's thm that $\mathrm{GKdim}(M) \geq n$.

— Bernstein / Gelfand ('68), Atiyah ('68) resolution of singularities (Hironaka). New pf in '72

Let $f[y_1, \dots, y_n] \in \mathbb{R}[y_1, \dots, y_n]$ be polynomial, non-neg. on interior of open, conn. $\Omega \subseteq \mathbb{R}^n$, $= 0$ on the boundary.

Given $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, then define $f_\Omega(\lambda) : \mathbb{R}^n \to \mathbb{C}$ (continuous)

$$f_\Omega(\lambda)(a) = \begin{cases} f(a)^\lambda & \text{if } a \in \Omega \\ 0 & \text{otherwise} \end{cases}.$$

Gelfand asked whether $f_\Omega(\lambda)$ can be extended to a meromorphic function on whole complex plane.

That is, Let $C_0^\infty(\Omega)$: smooth, comp. supp. (cx-valued) functions on Ω

$\mathcal{D}$: distributions on $C_0^\infty(\Omega)$ - continuous linear functionals w.r.t. $\|g\|_N$ family of semi norms

with $\|g\|_N = \max \{ |D^\alpha g(y)| \mid y \in \Omega, |\alpha| = \alpha_1 + \dots + \alpha_n \leq N \}$

and here $D^\alpha := \left(\frac{\partial}{\partial y_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial y_n}\right)^{\alpha_n}$

We say $\alpha : \Lambda \text{ : open set of } \mathbb{C} \to D$ is <u>analytic</u> if $\lambda \to \langle \alpha(\lambda), g \rangle$ is analytic function for each $g \in C_0^\infty(\Omega)$

For us: $\lambda \mapsto \langle f_\Omega(\lambda), g \rangle = \int_\Omega f(y)^\lambda g(y)\,dy$

By abuse, one can speak of $\lambda \mapsto f_\Omega(\lambda)$, $\Lambda \to D$ as <u>analytic</u>.

For $Re(\lambda) > 0$, we have

$$\frac{d}{d\lambda} \langle f_\Omega(\lambda), g \rangle = \int_\Omega \log(f(y))\, f(y)^\lambda\, g(y)\,dy$$

Extend this via functional equation (via Bernstein polynomial mentioned at the end of J. Goes' talk.)

Some special cases are easy: e.g. Gamma function: $f(y) = y$. (Test function: e^{-t})

e.g. $f(y) = y^2$. Do integration by parts twice for any test function g:

$$(2\lambda+1)(2\lambda+2) \int (y^2)^\lambda g(y)\,dy = \int (y^2)^{\lambda+1} g''(y)\,dy$$

use this to successively continue $\lambda \mapsto f_\Omega(\lambda)$ and obtain poles

at $\{ -1/2, -1, -3/2, -2, \dots \}$

<u>Rough idea</u>: view analytic functions from $\{Re(\lambda) > 0\} \to D$ as module over Weyl algebra A_n.

$K = \mathbb{C}(z)$ z: transcendental ,

$K(y_1, \dots, y_n)$: field of rational functions , which is $A_n(K)$-module

with $\frac{\partial}{\partial y_i}$ acting by quotient rule.

with submodule $K[y_1, \dots, y_n, f^{-1}] = \left\{ g/f^m \;\middle|\; \begin{matrix} \text{some } m \geq 0 \\ g \in K[y] \end{matrix} \right\}$.

Consider the free $K[y, f^{-1}]$-module $K[y, f^{-1}] \cdot F =: M$

(think: generator F represents f^λ.)

(could use action on $K[y, f^{-1}]$ to turn M into $A_n(K)$ module.
of $A_n(K)$

But want action of $\frac{\partial}{\partial y_i}$ on F to mimic $\frac{\partial (f^\lambda)}{\partial y_i}$ (i.e. chain rule)

so we set $$\frac{\partial}{\partial y_i} \cdot (aF) = \frac{\partial a}{\partial y_i} F + z\, \underbrace{a \frac{\partial f}{\partial y_i} f^{-1}}_{\in K[y, f^{-1}]} F$$

$a \in K[y, f^{-1}]$

with indeterminate z a stand-in for λ.

check this satisfies
$$\left[\frac{\partial}{\partial y_i}, y_j \right] = \delta_{ij}$$

Want to show $\exists$ integer $r \geq 0$, $q \in A_n(K)$ s.t.

$$q \cdot (f^{-r} F) = f^{-r-1} F.$$

which follows from fact that M (as left $A_n(K)$-module) has finite length

If we can show this, then the ascending chain:

$$A_n(K) \cdot F \subseteq A_n(K) \cdot (f^{-1}F) \subseteq \cdots \quad \text{must stabilize.}$$

Key lemma: finite length follows from having filtration compatible with Bernstein filtration s.t. $\dim_K(M_r)$ doesn't grow too fast in r.

Exhibit such a filtration of M by setting $M_r := \{ g f^{-r} F \mid g \in K[y]$ has $\deg(g) \leq (d+1)r \}$

where $d := \deg(f)$.

(so M_r is just elts. of form $g f^{-r}$ with total degree $\leq r$.)

with $\dim_K(M_r) = \binom{(d+1)r+n}{n}$ = # of monomials in vars $y_1, \dots, y_n$ with deg. $\leq (d+1)r$

pf of Thm: Let r be integer for which

$$q \cdot (f^{-r}F) = f^{-(r+1)}F$$

Let $\omega = z - r$, $G := f^{-r}F$. $K = \mathbb{C}(\omega)$

N: $A_n(K)$-submodule $K[y, f^{-1}] \cdot G$ with action defined so

$$\frac{\partial}{\partial y_i} \cdot G = \omega \frac{\partial f}{\partial y_i} f^{-1} G \quad (\text{so } \omega, G \text{ play roles of } z, f \text{ before})$$

$\mathbb{C}_s$: half open plane $\{ \lambda \mid \mathrm{Re}(\lambda) > s \}$

$\mathcal{S}$: $\{ \phi \mid \phi : \mathbb{C}_s \to \mathcal{D}$ analytic for some $s \in \mathbb{R} \} / \sim$: identify if agree on some half plane

initially just a $\mathbb{C}$-vector space. Show $\mathcal{S}$ made into $A_n(K)$-module.

To make $\mathcal{S}$ into $A_n(K)$-module define:

(i) $(\omega \circ \phi)(\lambda) = \lambda\, \phi(\lambda)$

(ii) $\left(\frac{\partial}{\partial y_i} \circ \phi\right)(\lambda) = \frac{\partial}{\partial y_i}(\phi(\lambda))$ (i.e. partial deriv. of distribution ϕ)

(iii) $(y_i \circ \phi)(\lambda) = (y_i \phi)(\lambda)$

Given f_Ω with $\langle f_\Omega(\lambda), g\rangle = \int_\Omega f^\lambda g \, dy$ on $\mathbb{C}_s$,

define $f^{-1} f_\Omega \in \mathcal{S}$ with domain $\mathbb{C}_{s+1}$ by

$(f^{-1} f_\Omega)(\lambda) := f_\Omega(\lambda - 1)$. which is reasonable since

$$\int_\Omega f^{-1} f^\lambda \cdot g \, dy = \int_\Omega f^{\lambda-1} \cdot g \, dy = \langle f_\Omega(\lambda-1), g\rangle.$$

for any $\lambda \in \mathbb{C}_s$, $s \geq 1$.
any $g \in C_c^\infty(\Omega)$.

Further $\frac{\partial}{\partial y_i} f_\Omega(\lambda) = \lambda \frac{\partial f}{\partial y_i} f_\Omega(\lambda - 1)$ for $\lambda \in \mathbb{C}_s$, $s \geq 1$.

<u>Goal</u>: Extend domain of f_Ω from C_0 to C_{-1}, which will be accomplished by extending $f^{-1} f_\Omega$ from C_1 to C_0. (though extension won't be in $\mathcal{S}$ as it may have poles)

Key is $A_n(K)$-module homom.

$$\phi : N = K[y, f^{-1}] \cdot G \longrightarrow K[y, f^{-1}] \cdot f_\Omega \subseteq \mathcal{S}$$

$$\phi(c(y, f^{-1}, \omega)\, G)(\lambda) = c(y, f^{-1}, \lambda)\, f_\Omega(\lambda)$$

where to verify it is module homom: check $\phi(\frac{\partial}{\partial y_i} \circ G) = \frac{\partial}{\partial y_i} \phi(G)$

$$\phi(\frac{\partial}{\partial y_i} \circ G)(\lambda) = \phi(\omega \frac{\partial f}{\partial y_i} f^{-1} G)(\lambda) = \lambda \frac{\partial f}{\partial y_i} f^{-1} f_\Omega(\lambda)$$

$$= \lambda \frac{\partial f}{\partial y_i} f_\Omega(\lambda - 1) = \frac{\partial f}{\partial y_i} f_\Omega(\lambda) = \frac{\partial}{\partial y_i} \phi(G)(\lambda).$$

for $\lambda \in \mathbb{C}_1$

Let $q \in A_n(K)$ s.t

$q \circ (f^{-r} F) = f^{-(r+1)} F$. Write $q = \frac{a}{b}$ with

$a \in A_n(\mathbb{C}[\omega])$

so that $b \circ (f^{-1} G) = a \circ G$ $\quad b(\omega) := b \in \mathbb{C}[\omega]$

So $b(\lambda) f_\Omega(\lambda - 1) = b(\lambda)(f^{-1} f_\Omega)(\lambda) = \phi(b(\omega) f^{-1} G)(\lambda)$

$$= \phi(a \circ G)(\lambda) = a f_\Omega(\lambda)$$

$\Rightarrow$ $\langle b(\lambda) f_\Omega(\lambda - 1), g \rangle = \langle a f_\Omega(\lambda), g \rangle = \langle f_\Omega(\lambda), a^\# g \rangle$

for all $\lambda \in \mathbb{C}_1$, $g \in C_0^\infty(\Omega)$, $a^\#$: adjoint of diff. op. a

so extend f_Ω meromorphically to $\mathbb{C}_{-1}$ by setting

$$\langle f_\Omega(\lambda - 1), g \rangle = \frac{1}{b(\lambda)} \langle f_\Omega(\lambda), a^\# g \rangle$$

repeating gives desired analytic continuation, where poles given by zeros of $b(\lambda)$ in arithmetic progression over all integers.