$G : \mathbb{R}^n \to \mathbb{R}^n$, class C^r, $r > 2$. $G(0) = 0$.

$$\dot{X} = G(X)$$

$X = 0$ is unique sol'n satisfying $X(0) = 0$.

The ~~singular~~ fixed point $X = 0$ is asymptotically stable if

(1) Given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. for every p_0 with $|p_0| < \delta$ the solution ϕ to PDE with ~~the~~ initial condition $\phi(0) = p_0$ can be extended to the half line $t > 0$ s.t. $|\phi(t)| < \epsilon$ $\forall\ t > 0$.

(2) $\exists\ \eta > 0$ s.t. $\lim_{t \to \infty} \phi(t) = 0$ $\forall\ \phi$: solutions of PDE s.t. $\phi(0) < \eta$.

—

Thm A: $n \times n$ matrix $\in Mat_n(\mathbb{R})$. Then 0 is asymptotically stable singular point of $\dot{X} = AX \Leftrightarrow$ e-values of A have negative real part.

Lyapunov showed this extends to linearized systems:

$$\dot{X} = \underbrace{JG(0)}_{\text{Jacobian of } G \text{ at } 0} \cdot X$$

Thm: If real part of e-values of $JG(0)$ are negative, then 0 is asymptotically stable point of PDE.

Globally asymptotically stable if η (in definition of asymp. stable) may be taken to be ∞.

In linear system, if 0 is asymptotically stable, then globally asymp. stable.

Conjecture (Markus-Yamabe, 1960)

$X = 0$ is globally asymptotically stable for $\dot{X} = G(X)$

if, for every $P \in \mathbb{R}^n$, ~~then~~ $X = 0$ is asymptotically stable

for the linear system $\dot{X} = JG(P)\,X$

$\underline{n=2}$: True. Gutierrez '93. (G: polynomial : Meisters/Olech '88)

$\underline{n \geq 4}$: False Barbanov '88

In $n=2$ case, ~~can~~ then $\dot{X} = JG(P)\,X$ has $X=0$ as asymptotically stable $\iff$ $JG(P)$ has $\det(JG(P)) > 0$
$\operatorname{tr}(JG(P)) < 0$

$\mathcal{F}$: class of C^1 maps $F: \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

(a) $F(0) = 0$

(b) $\det(JG(P)) > 0 \quad \forall P \in \mathbb{R}^2$

(c) $\operatorname{tr}(JG(P)) < 0 \quad \forall P \in \mathbb{R}^2$

Link to Jacobian conjecture (?).

Prop.: Suppose origin is globally asymptotically stable pt. of $\dot{X} = F(X)$ for every polynomial map $F \in \mathcal{F}$. Then polynomials in $\mathcal{F}$ are injective

(so maps in $\mathcal{F}$ invertible)

Proposition: Suppose the origin $X=0$ is a globally asymptotically stable point of $\dot{X} = F(X)$, for every polynomial $F \in \mathcal{F}$. Then the polynomial maps in $\mathcal{F}$ are injective.

pf: Suppose $F \in \mathcal{F}$ not injective. So $\exists$ P_1, P_2 s.t. $F(P_1) = F(P_2) = Q$. Consider $\dot{X} = H(X)$ where

$H(X) = F(X+P_1) - Q$. Two equilibrium points: $X=0$, $X = P_2 - P_1 \neq 0$.

so $X=0$ can't be globally asymptotically stable. Is $H \in \mathcal{F}$?

Yes, since $J(H)(X) = JF(X+P_1)$. So this gives contradiction.

(would state this as corollary)
but we'll use it in course of proving main theorem.

Main
<u>Thm</u> (Meisters/Olech) Let $F \in \mathcal{F}$. Then $X = 0$ is a globally asymptotically stable point of the system $\dot{X} = F(X)$.

Will follow from two results:

<u>Thm</u> A: $F: K^n \to K^n$ polynomial, if $\det J(F) \neq 0$ everywhere in K^n then $\exists$ b: pos int. such that $F^{-1}(P)$ has at most b^n points for every $P \in K^n$.

(Give D-module proof due to van den Essen '91).

<u>Thm</u> B: $F \in \mathcal{F}$. <u>if</u> $\exists$ positive consts. ρ, r such that $|F(x)| \geq \rho$ whenever $|x| \geq r$, then $X = 0$ is a globally asymptotically stable point of the system $\dot{X} = F(X)$.

(Application of Green's thm, due to Olech '63. Pure analytic proof)

proof of main theorem: Since $F \in \mathcal{F}$, then by theorem A,

$$\sup \{ |F^{-1}(Y)| : Y \in \mathbb{R}^2 \} = k : \text{const} < \infty .$$

Let p be point for which maximum is attained, and label

$F^{-1}(P) = \{ Q_1, \dots ; Q_k \}$. By inverse function thm, we may

choose nbhds V_i of Q_i s.t. $F : V_i \to B_\rho(P)$ is a diffeomorphism.

Ball of radius ρ centered at P

Also pick V_i disjoint.

Then $F^{-1}(B_\rho(p)) = V_1 \cup \dots \cup V_k$.

(if $\exists\, W \notin V_1 \cup \dots \cup V_k$ s.t. $F(w) \in B_\rho(p)$

then $F(Y_i) = F(w)$ $\forall$ $i = 1, \dots, k$, contradicting maximality.)

pick $r' > 0$ so that $V_1 \cup \dots \cup V_k \subseteq B_{r'}(0)$. Then

$$|F(X) - p| \geq \rho \quad \text{if} \quad |X| \geq r' \qquad (*)$$

Set $G(x) = F(X + Q_1) - P$. Then $G(X)$ satisfies hypotheses of

thm B. (i.e. $X = 0$ is asymptotically stable point of $\dot{X} = G(X)$.)

By earlier proposition, this implies G injective

But $G(Q_i - Q_1) = F(Q_i) - P = 0 = G(0)$ for $i = 1, \dots, k$

so if G injective, then $k = 1$.

$\Rightarrow$ $F^{-1}(P)$ has at most one point for any $p \in \mathbb{R}^2$, so $p = 0$ attains max

$(F \in \mathcal{F} \Rightarrow F(0) = 0)$

and hence $(*)$ valid for $p = 0$. Thus thm follows from thm B.

$F : K^n \to K^n$: polynomial $F = (F_1, \dots, F_n)$ and write

$\Delta(x) = \det JF(x)$. Suppose $\Delta(x) \neq 0$ for every $x \in K^n$.

Given $g \in K[X, \Delta^{-1}]$ consider the derivation D_i given by

$$D_i(g) = \Delta^{-1} \det J(F_1, \dots, \underset{\substack{\uparrow \\ i^{th} \text{ pos.}}}{g}, \dots, F_n)$$

with $[D_i, F_j] = \delta_{ij}$, $[D_i, D_j] = 0$. $\forall$ i, j

Let $M(F)$ be the D-module (A$_n$-module) defined as $K[X, \Delta^{-1}]$ with action

$x_i \cdot g = F_i \cdot g$ (not hard to check this well-defined)

$\partial_i \cdot g = D_i(g)$ $g \in K[X, \Delta^{-1}]$

Key Lemma (in pf of thm A) : As A_n-module, $M(F)$ is holonomic with multiplicity $\leq 2^n (2nd+1)^n$ where $d = \max_i \deg(F_i)$.

(analogous to earlier result which was key in proving existence of Bernstein polynomial. $\partial_i(f/p^r) := \partial_i(f) \cdot p^r - f\, \partial_i(p^r) / p^{2r}$)

multiplicity bounds length, and finite length gave rise to functional equation.

steps : ① $\Gamma_\nu := \{ g \cdot \Delta^{-2\nu} \in K[X, \Delta^{-1}] : \deg(g) \leq 2\nu(2nd+1) \}_{\nu \in \mathbb{N}}$

is a filtration.

② Γ_ν as K-vector space has dimension $\dim_K \Gamma_\nu \leq \dfrac{2^n (2nd+1)}{n!} \nu^n +$ smaller order terms

(counting polynomials of given degree)

(10.3.1)

<u>Lemma</u>: M left A_n-module with filtration Γ w.r.t. Bernstein filtration of A_n. Suppose $\exists$ c_1, c_2 s.t. for $j \gg 0$,

$$\dim_K \Gamma_j \leq c_1 \left(j^n/n!\right) + c_2 (j+1)^{n-1}$$

$\rightarrow$ M holonomic A_n-module with multiplicity $\leq c_1$

(pf. of lemma is that if Hilbert poly growing at most like this, then must be of deg $\leq n$.)

+ Bernstein's lower bound.

<u>proof of theorem A</u> : (where we finally make use of Kashiwara's thm.)

To any $P \in K^n$, consider polynomial $F-P$ (assume $\det J(F) \neq 0$ everywhere $\Rightarrow$ $\det J(F-P) \neq 0$ everywhere)

Let $M(P) := M(F-P)$, the set $K[X, \Delta^{-1}]$ with A_n-module structure given as above (depends on P, in that $x_i \cdot g = (F_i - P_i) \cdot g$)

Previous lemma ensures $M(P)$ is holonomic, with multiplicity $\leq b$

By Kashiwara's thm, the embedding $\iota : \{pt\} \longrightarrow \{pt\} \times X$, $\ast \longmapsto (\ast, 0)$

induces map ι^* on D-modules with

$\iota^* : M \longmapsto M / (x) M$, preserving holonomicity. with mult. $\leq b$.

so $M(P) \Big/ \sum_i x_i \cdot M(P) = M(P) \Big/ \sum_i (F_i - P_i) \cdot M(P)$

is holonomic with multiplicity $\leq b$ as well,

so as vector space over K, has dimension $\leq b$.

$\Rightarrow$ $\{1, x_1, x_1^2, \ldots, x_1^b\}$ linearly dependent

$\Rightarrow$ $\exists$ $g_1 \in K[x_1]$, $\deg \leq b$, and positive integer r s.t.

$$\Delta^r g_1(x_1) \in \sum_i (F_i - P_i) M(P)$$

Given $Q = (Q_1, \dots, Q_n) \in K^n$ s.t. $F(Q) = P$, then

$\Delta(Q)^r\, g(Q_1) = 0 \Rightarrow g(Q_1) = 0$ since $\Delta \neq 0$ on K^n.

so there are at most b possibilities for Q_1. Repeating, we have at most b^n choices for Q. (Since P arbitrary, result follows)