Definition (Duflo) A K-algebra A is almost commutative if there exists a filtration $\underbrace{A_0 \subseteq A_1 \subseteq \cdots}_{\mathcal{A}}$ such that

① $A_0 = K$

②* A_1 is finite dimensional, A is generated as an algebra by A_1

③ $gr_{\mathcal{A}}(A) = \bigoplus_{i=0}^{\infty} A_i / A_{i-1}$ is commutative.

Proposition: Let A be an almost comm. K-algebra w.r.t. $\mathcal{A} = \{A_i\}$. Then

① $gr_{\mathcal{A}}(A)$ is fin. gen. commutative noetherian algebra

② A is left and right noetherian.

pf: As algebra $gr_{\mathcal{A}}(A)$ is generated by A_1/A_0 and each A_i/A_{i-1} is spanned by words of length i using the (finitely many) basis elts of the vector space A_1/A_0

$\Rightarrow$ $\exists$ graded K-algebra homom. $S \twoheadrightarrow gr_{\mathcal{A}}(A)$. But S noetherian by Hilbert Basis Thm. So $gr_{\mathcal{A}}(A)$ is as well.

symmetric algebra on A_1/A_0

polynomial ring in basis vectors for A_1/A_0

If graded module over graded alg. is Noetherian, then module over algebra is Noetherian

commutative, noetherian algebras.

$\mathfrak{g}$: finite dim'l Lie algebra / K

$\downarrow$

$\mathcal{U}(\mathfrak{g})$: univ. env. algebra $\overset{\text{def}}{=}$ tensor algebra $:= \bigoplus_{k=0}^{\infty} (\underbrace{\otimes^k \mathfrak{g}}_{\mathfrak{g}^k})$ $/ \langle X \otimes Y - Y \otimes X - [X,Y] \mid X, Y \in \mathfrak{g} \rangle$

Has filtration $\{\mathcal{U}_i\}$ with

$\mathcal{U}_0 = K, \quad \mathcal{U}_n = \mathfrak{g}^n, \ n > 0.$

Poincaré - Birkhoff - Witt theorem:

- $gr(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g})$: symmetric alg. on $\mathfrak{g}$.
- Given ordered basis of $\mathfrak{g}$, $\{x_1, x_2, \ldots\}$, then

  "ordered monomials" $x_{i(1)} \cdots x_{i(m)}$ $\quad i(1) \le \cdots \le i(m)$, and unit 1,

  form a basis of $\mathcal{U}(\mathfrak{g})$.

(In particular, $gr(\mathcal{U}(\mathfrak{g}))$ is commutative with $GK\dim(\mathcal{U}(\mathfrak{g}))$
$= GK\dim(S(\mathfrak{g})) = \dim_K \mathfrak{g}$)

Moreover, if I : ideal of $\mathcal{U}(\mathfrak{g})$,

then $\exists$ induced filtration on $\mathcal{U}(\mathfrak{g})/I$, with $gr(\mathcal{U}(\mathfrak{g})/I) \cong S(\mathfrak{g})/gr(I)$

a Noetherian, commutative K-algebra with G-K dim. an integer.

After defining almost comm. algebra...

[Point out relationship between $GK\dim(M) \geq K\dim(M)$

M: f. gen. (left-) A module. A almost commutative]

Basic idea: Know any f. gen. commutative K-algebra is a f. gen. module for $K[x_1, \ldots, x_r]$ with $r =$ classical Krull dim.

Noether normalization thm

But $GK\dim(A)$
$= GK\dim(K[\underline{x}])$
$= r.$

(= classical Krull dimension
= length of longest proper chain of prime ideals.)
(count inclusions, not ideals)

Theorem: An K-algebra ~~almost commutative~~ almost commutative $\Leftrightarrow$ it is the homomorphic image of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$: $\mathfrak{g}$ is fin. dim'l Lie algebra $/K$

pf: Suppose A is almost commutative w.r.t. filtration $\{A_i\}$.

Then by assumption, A_1/A_0 is commutative so for $x, y \in A_1$

$$\underbrace{[x + A_0][y + A_0] - [y + A_0][x + A_0]}_{[(xy - yx) + A_1] \in A_2/A_1} = 0 \qquad \Rightarrow \quad xy - yx \in A_1.$$

$\Rightarrow$ f. dim'l vector space A_1 has Lie algebra structure. $\mathfrak{g}$

By universal prop. of $\mathcal{U}(\mathfrak{g})$ + A is generated by $A_1 = \mathfrak{g}$,

then canonical embedding $\mathfrak{g} \longrightarrow A$ can be extended to a

unique K-algebra homom. of $\mathcal{U}(\mathfrak{g})$ onto A.

$(\Leftarrow)$ $\phi : \mathcal{U}(\mathfrak{g}) \longrightarrow A$ K-algebra homom.

Let $\{\mathcal{U}_n\}$ be the usual filtration of $\mathcal{U}(\mathfrak{g})$.

So that $\phi(\mathcal{U}_n) =: A_n$ gives a (discrete) filtration of A.

$A_0 = \phi(\mathcal{U}_0) = \phi(K) = K$

And since $\mathcal{U}_1$ generates $\mathcal{U}(\mathfrak{g})$, finite dimensional,

so does $\phi(\mathcal{U}_1) =: A_1$

For each n, ϕ induces map on $\mathcal{U}_n / \mathcal{U}_{n-1} \longrightarrow A_n / A_{n-1}$

combine to give graded k-module map from $gr(\mathcal{U}(\mathfrak{g})) \longrightarrow gr(A)$
$gr(\phi)$

By PBW thm, $gr(\mathcal{U}(\mathfrak{g}))$ is symmetric algebra on vector space $\mathfrak{g}$

So done if we can show $gr(\phi)$ is a ring homom. (easy check:

for this $\bar{u} = [u + \mathcal{U}_{m-1}]$, $\bar{v} = [v + \mathcal{U}_{n-1}] \in gr(\mathcal{U}(\mathfrak{g}))$

u: deg m, v: deg. n
(non-zero, homog.)

$$\bar{u} \cdot \bar{v} = [uv + \mathcal{U}_{m+n-1}]$$

$$\begin{aligned} gr(\phi)(\bar{u} \cdot \bar{v}) &= [\phi(uv) + \mathcal{U}_{m+n-1}] \\ &= [\phi(u)\phi(v) + A_{m+n-1}] \\ &= [\phi(u) + A_{m-1}][\phi(v) + A_{n-1}] = gr(\phi)(\bar{u}) \cdot gr(\phi)(\bar{v}). \end{aligned}$$
)

Example: A_n is the homomorphic image of n^{th} Heisenberg algebra

(Lie alg. with gens $x_1, \dots, x_n, y_1, \dots, y_n, z$ with $[x_i, y_i] = z$
other brackets $= 0$.)

(Just send $z \mapsto 1$.)

Lie algebra exponentiates to matrix Lie group : $\begin{pmatrix} 1 & \underline{x} & z \\ & I_n & \underline{y} \\ & & 1 \end{pmatrix}$

Mention Stone-von Neumann thm.

Krull dimension v. GK dimension.

$\log_n := \log d_M(n) / \log n$.

$$\mathrm{GK\,dim}(M) = \overline{\lim} \left(\log_n d_M(n)\right) \qquad d_M(n) = \dim\left(\bigoplus_{i=-n}^{n} M_i\right) \quad \text{(integer grading)}$$

M: f. gen. left A-module, A: fin. graded K-algebra

(without grading, still do something similar ...)

definition from '67 paper by Rentschler and Gabriel

---

classical Krull dimension (for commutative rings): length of maximal chain of prime ideals

---

generalization to modules over non-commutative rings R (or better: (not nec. commutative)), defined inductively for all ordinals α

K_{-1}: class of R-modules consisting of 0-module.

K_α: R-modules s.t. for every countable descending chain of submodules

$$M_0 \geq M_1 \geq M_2 \geq \cdots$$

$$M_i / M_{i+1} \in \bigcup_{\beta<\alpha} K_\beta \quad \text{for all but finitely many } i.$$

Least such α to which M belongs: Krull dimension.

(If descending chain condition is satisfied $\Leftrightarrow$ M has Krull dim. 0.
M Artinian, $\neq (0)$.)

→ Short exercises:
R left-Noetherian, then left Krull dim defined

Further: (left) Krull dim $\geq$ classical Krull dim.

Example: $A_1(K)$. Classical Krull dim. $= 0$ (since A_1 simple)

in this case

But $A_1(K)$ not Artinian, so

$$\ell.K.\dim(A_1) > 0.$$

(In fact it is equal to 1)

and $A_n(K)$ has $\ell.K.\dim(A_n) = n$.

(one way to show: prove thm. on relation btw. GK dim and Kdim.)

Key Lemma on dimension : A : almost comm. K-algebra. $m \geq 0$, integer.

Suppose every fin. gen. A module of Krull dim. m has GK dimension $m+r$, $r \geq 0$ integer. Then any f. gen. A module M with $\mathrm{Kdim}(M) \geq m$ has $\mathrm{GKdim}(M) \geq \mathrm{Kdim}(M) + r$.

Corollary (setting $m = r = 0$, so condition trivially satisfied) For any non-zero fin. gen. A-module M,

$\mathrm{GKdim}(M) \geq \mathrm{Kdim}(M)$

pf of lemma : By induction on Kdim.

Suppose true for all fin-gen A-modules N with $m \leq \mathrm{Kdim}(N) < \alpha$.

Let M have $\mathrm{Kdim}(M) = \alpha$. $\Rightarrow$

$\exists$ infinite descending chain $M = M_0 \supseteq M_1 \supseteq \cdots$ of submodules s.t.

$\mathrm{Kdim}(M_i/M_{i+1}) \geq \alpha - 1$. By inductive hypothesis $\mathrm{GKdim}(M_i/M_{i+1}) \geq \alpha - 1 + r$

why is it an integer?

$\Rightarrow$ $\mathrm{GKdim}(M) \geq \alpha - 1 + r$ i.e. $\mathrm{GKdim}(M) \geq \alpha + r$.

Consequence of exactness of GKdim.

used by Smith to show $\mathrm{Kdim}(\mathcal{U}(\mathfrak{sl}_2)) = 2$.

Prop : M: non-zero fin-gen. module over fin-gen commutative K-alg. A, then

$\mathrm{GKdim}(M) = \mathrm{Kdim}(M)$.

pf : Use exactness to reduce to A. For A, follows from fact that

$\mathrm{GKdim}(A) = \mathrm{cl.Kdim}(A)$. //

More generally, "D-module" refers to the sheaf of differential ops.

on smooth, affine variety X (as a subsheaf of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$)

The associated graded sheaf (with respect to order filtration : $\mathcal{D}_X^i$: diff. ops. of order $\leq i$)

$gr(\mathcal{D}_X) \simeq S(T_X)$: symmetric algebra of tangent bundle

$\Rightarrow$ $\mathcal{D}_X$ is almost commutative. (*)

(*) Issue with this filtration is that resulting subspaces are no longer finite-dim'l.

so Gelfand-Kirillov dimension is not so useful.

Could instead define $\dim(M) := \mathrm{Kdim}(gr(M))$