

9. Dimension & Multiplicity

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Oct. 8. 2012

Let M be a F.G. A_n -module.

\mathcal{P} : good filtration.

$$\dim_k(\Gamma_t) = \chi(t, \Gamma, M) \quad \checkmark \text{ polynomial. } t \gg 0.$$

$$d(M) = \deg(\chi(t, \Gamma, M)) \quad \chi \in \mathbb{Q}[t].$$

$$m(M) := (\text{leading term}) \cdot \left(\frac{\dim_k(M)}{d(M)!} \right)$$

Df: (1) Let $p \in \mathbb{Q}[t]$, p is ~~per~~ numerical.

$$\Leftrightarrow p(s) \in \mathbb{Z} \quad \text{for } s \gg 0.$$

(2) Let $f: R \rightarrow R$, $\Delta f(z) = f(z+1) - f(z)$
 $z \in R$.

$$(3) \quad \binom{t}{r} := \frac{t(t-1)\cdots(t-r+1)}{r!} \quad \binom{t}{0} := 1.$$

Lemma 1 ① $\Delta \binom{t}{r} = \binom{t}{r-1}$... $\binom{t}{-1} = 0$ 9-2

Δ is \mathbb{R} -linear

$$\Delta(f \pm g) = \Delta f \pm \Delta g$$

② Let p be a numerical poly. of deg k .

$$\exists c_0, c_1, \dots, c_k \in \mathbb{Z}.$$

$$p(t) = \sum_{i=0}^k c_{k-i} \binom{t}{i}.$$

③ let $f: \mathbb{Z} \rightarrow \mathbb{Z}$, s.t.

$$\Delta f(s) = q(s), \quad q \in \mathbb{Q}[t], \quad t \gg 0$$

then $f(s) = p(s)$, $s \gg 0$ for some $p \in \mathbb{Q}[t]$.

Proof: ① clear ② induction on k . ③ ~~Can~~ use ② and Δ and ~~use~~ Δ .

Th. M be a $\overbrace{\text{F.G. Module}}^{\text{graded}}$ / $k[x_1, \dots, x_n]$

$$s \in \mathbb{Z}_{\geq 0}, \quad \sum_{i=0}^s \dim_k(M_i) \quad \exists \chi \in \mathbb{Q}[t], \text{ s.t.}$$

$\chi(s) //$ when $s \gg 0$. | χ is called Hilbert polynomial.

Pf.

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M is finite dim. vector space

For $s \gg 0$, ~~M_i~~ $\dim(M_i) = 0$ $\chi(s) = \text{co} \in \mathbb{Z}$

Suppose that the statement is true for $k[x_1, \dots, x_{n-1}]$.

Consider $k[x_1, \dots, x_n]$.

Take M . Let $i \geq 0$.

$\phi_i : \cancel{M_i} \rightarrow \cancel{M_{i+1}}$ multiply by x_n . $\deg(x_n) = 1$

$$x_i \rightarrow M_i \rightarrow M_{i+1} \rightarrow L_{i+1} \rightarrow 0$$

$x_n(k_i) = 0$, $x_n(L_{i+1}) = (0)$. k_i is kernel
 L_{i+1} cokernel.

So k_i, L_{i+1} are modules over $k[x_1, \dots, x_{n-1}]$.

$$K = \bigoplus k_i, \quad L = \bigoplus L_i$$

$$\text{So, } s \gg 0, \sum_{i=0}^s \dim k_i = \chi_1(s), \quad \sum_{i=0}^s \dim L_i = \chi_2(s).$$

$$\text{Also, } \dim k_i - \dim k_{i+1} + \dim M_{i+1} - \cancel{\dim M_i} = 0,$$

$$f(s) := \sum_{i=0}^s \dim(M_i)$$

$$\chi_1(s) + \Delta f(s) - \chi_2(s) = 0, \quad s \gg 0.$$

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$$\Delta f(s) = -\chi_1(s) + \chi_2(s)$$

By ~~Lea~~ lemma 1.13), $f(s) = P(s)$.

II. dim and multiplicity of A_n -modules.

Df. M : F.G. A_n -module.

$\Gamma \ni$: good filtration. w.r.t. B-fil.

$$\dim_K \Gamma_t = \sum_{i=0}^t \dim_K (\Gamma_i / \Gamma_{i-1}).$$

$gr^T(M)$ is a S_n -module, with

$$S_n = K[X_1, \dots, X_n, \bar{\partial}_1, \dots, \bar{\partial}_n].$$

The Hilbert polynomial of $gr^T(M)$

$$\chi(s, \Gamma, M) = \deg \chi,$$

$$m(M) \cdot (\text{leading coeff. of } \chi) \cdot d(M)!$$

Prop. Γ, Ω good filtrations on M .

Then $\deg(\chi(s, \Gamma, M)) = \deg(\chi(s, \Omega, M))$.

By Prop 8.3.2 $\exists k$, s.t. $\forall i \gg 0$, for all

$$\Omega_{i-k} \subset \Gamma_i \subset \Omega_{i+k},$$

So $\dim(\Omega_{i-k}) \leq \dim(\Gamma_i) \leq \dim(\Omega_{i+k})$.

$\exists s_0$, s.t. $\forall s \geq s_0$.

$$\chi(s-k, \Omega, M) = \frac{\sum_{i=0}^s \dim(\Omega_{i-k})}{s} \leq \chi(s, \Gamma, M) \leq \chi(s+k, \Omega, M).$$

So their degrees are the same.

* Also ^{work} for leading term.

Example $M = A_n$, B be the filtration.

$$B_s = \langle x^\alpha \partial^\beta \mid |\alpha| + |\beta| \leq s \rangle$$

$$\dim B_s = \binom{2n+s}{2n}.$$

Leading term $\frac{t^{2n}}{(2n)!}$

$d = 2n$
 $m = 1$

$$(2) \quad M = K[x_1, \dots, x_n]$$

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$$d(M) = n, \quad M(M) = 1$$
