10.10

$d(M)$ is the degree of $gr^{\Gamma}(M)$.

Prop. $d(M) = d(M_\sigma)$. σ: auto. of A_n.

Let $V = \text{span}_K \langle 1, \text{finite set} \rangle \subset A_n$.

Γ: good filtration $\Gamma_0 \neq 0$.

$$U_k = \begin{cases} K & k=0 \\ V^k & k>0. \end{cases}$$

$\Omega_k = U_k \cdot \Gamma_0$.

$\delta(M,V) = \inf \{ \nu \mid \dim_K \Omega_s \leq s^\nu \text{ for } s \gg 0 \}$

this would be $d(M)$

Suppose $V \subset B_r$, then $V^k \subset B_{rk}$.

$\Omega_k \subset B_{rk} \cdot \Gamma_0$

$\Gamma_i = B_i \cdot \Gamma_0$. Assume.

(because $\Gamma_{i+k} = B_i \Gamma_k$ when k large, we let $\Gamma_0' = \Gamma_k$. . . .)

So $\dim_K \Omega_k \leq \dim_K \Gamma_{rk}$

So $\chi(k, \Omega, M) \leq \chi(rk, \Gamma, M) \leq (\gamma k)^{\delta(M, B_1)}$

~~$V = B_1$~~, $\Rightarrow \delta(M, V) \leq \delta(M, B_1)$

Similarly, $\delta(M, B_1) \leq \delta(M, V)$. So . . .

B_σ is a filtration on ~~...~~ A_n, pull it back.

It ~~ast~~ acts on Γ. ~~...~~ may not be the same as B,

but both are good. So the polynomial leading term of the Hilbert

should be the same.

III). $0 \to N \to M \to M/N \to 0$

Γ is good on M, Γ', Γ'' ~~on~~ good on N, M/N.

$$0 \to gr^{\Gamma'} N \to gr^{\Gamma} M \to gr^{\Gamma''}(M/N) \to 0$$

Then $\dim_k(\Gamma_i/\Gamma_{i-1}) = \dim_k(\Gamma'_i/\Gamma'_{i-1}) + \dim_k(\Gamma''_i/\Gamma''_{i-1})$.

For $s >> 0$

$$\chi(s, \Gamma, M) = \chi(s, \Gamma', N) + \chi(s, \Gamma'', M/N).$$

$$\Rightarrow d(M) = \max\{d(M/N), d(N)\}$$

(Leading term can't be negative)

$$m(M) = m(N) + m(M/N).$$

if $d(N) = d(M/N)$

<u>Prop.</u> $d(M) \leq 2n$. $\left(\overset{\text{note that}}{d(A_n) = 2n} \right)$

if M is F.G. A_n-module

<u>Prop</u> $M = A_n / I$ then $\dim(M) \leq 2n-1$

<u>Case 1</u>. $I = (a)$.

$$0 \to A_n \xrightarrow{\cdot a} A_n \to M \to 0$$

But both A_n have the same leading term.

they cancel.

~~Genel~~ <u>General</u>.

$$\to ? \to A_n/(a) \to A_n/I \to 0.$$

So . -

Thm: $d(M) \geq n$, $M \neq (0)$.

Lemma: M: F.G. A_n-module

Γ: good filtration.

$$B_i \longrightarrow \mathrm{Hom}_k(\Gamma_i, \Gamma_{2i})$$
$$\varphi: a \longmapsto \text{"}\underline{a\cdot}\text{"}$$

We claim φ is injective.

Let $a \in B_i$, $a \neq 0$, we need: $a\Gamma_i \neq (0)$.

Induction on i: $i=0$ clear. Suppose $i \leq k-1$ all right, ~~now~~ when $i = k$, Argue by Contradiction.

Find some ∂_j s.t. $[a, \partial_j] \neq 0$ then $[a, \partial_j] \in B_{k-1}$.

So $[a, \partial_j]\Gamma_{i-1} \overset{=0}{} \cancel{\subset a\partial_j \Gamma_{i-1} \text{ since } a\Gamma_{i-1} \subset a\Gamma_i = 0}$ since $a\Gamma_i = 0$.

but $[a, \partial_j]\Gamma_{i-1} \neq 0$, Cont!

Now if M is F.G. A_n-module,

Γ. good filtration.

By previous, $\dim(B_i) \le \dim(\mathrm{Hom}_k(\Gamma_i, F_{2i}))$

for $i \gg 0$.

so $\cancel{2}\,\chi(M) \overset{?}{\cancel{\ge}}\,\chi(A_n)$, $\square$.

(More or less..)