Thus every elt $x \in \tau L \cdot K_v$ is expressible as limit of elts in τL,

i.e. $x = \lim_{n \to \infty} \tau x_n$, $x_n \in L \ \forall n$

Since $\bar{v} \circ \tau = \bar{v} \circ \tau'$ then $\{\tau' x_n\} = \{\sigma \tau x_n\}$ converges in $\bar{v} \circ \tau'$ topology of $\tau' L \cdot K_v$

Call resulting limit σx.

Then $x \mapsto \sigma x$ is our desired isomorphism $\tau L \cdot K_v \to \tau' L \cdot K_v$.

and it leaves K_v fixed. Now extend to autom. $\tilde{\sigma}$ of $\underbrace{Gal(\bar{K}_v / K_v)}$.

Why is such an extension guaranteed? One alternative is to say equivalent/conjugate extensions require isom. of compositums $\tau L \cdot K_v \to \tau' L \cdot K_v$ in fixed alg. closure.

Then $\tau' L \cdot K_v / K_v$ is finite extn. so has unique abs. value extending one on K_v.

(As in Jacobson, B.A. II p. 585)

Fancy algebraic formulation: tensor products of vector spaces -

Have natural homomorphisms

$$L \otimes_K K_v \longrightarrow L_w = L \cdot K_v$$

$$a \otimes b \longmapsto ab \quad \text{(or maybe better } \tau(a) \cdot b \text{)}$$

to emphasize K-embedding

?

L as K-vector space now viewed as K_v-vector space (extension of scalars)

If we do this for all places w over v for L,

then have map

$$\phi : L \otimes_K K_v \longrightarrow \prod_{w|v} L_w$$

Proposition: If L/K separable, then ϕ is isomorphism.

proof of Proposition: Use earlier characterization with α primitive elt, f its minimal polynomial.

$$f(x) = \prod_{w|v} p_w(x) \quad \text{over } K_v[x]$$

view L as embedded in $\overline{K_v}$. (and L_w)

Let α_w: image of $\alpha \in L$ in $L_w \subseteq \overline{K_v}$.

thus we have commutative diagram:

$$\begin{tikzcd}
K_v[x]/(f) \arrow[r, "\sim", "\text{C.R.T.}"'] \arrow[d, "S"] & \prod_{w|v} K_v[x]/(p_w) \arrow[d, "l"] \\
L \otimes_k K_v \arrow[r, "\text{homom. above}"'] & \prod_{w|v} L_w
\end{tikzcd}$$

$x \mapsto \alpha$

since $L = k[x]/(f) = k[\alpha]$ and K_v extends scalars

then since $L_W \cong L \cdot k_v$ we have $L_W = K_v(\alpha_w)$. with $p_w(x)$ min poly. of α_w. so $[L_W : K_v] = \deg(p_w)$ "local degree"

by this discussion

But since top and sides are isom, bottom must be isom as well.

(explicitly sends $\alpha \otimes 1 \longmapsto \alpha_w$ for each $w|v$)

Corollary: $$[L:K] = \sum_{w|v} [L_W : K_v]$$

If $L|K$ separable, finite

if v: discrete, then since $[L_W:K_v] = e_w \cdot f_w$, where $e_w = [w(L^*) : v(K^*)]$, $f_w = [\lambda_{L_W} : \kappa_{K_v}]$

we have $$[L:K] = \sum_{w|v} e_w f_w$$

Better notation: $[\lambda_W : \kappa_v]$

pf of corollary is immediate from proposition: $[L:K] = \dim_K(L)$ ↑ i.e. as a K-vector space

$$= \dim_{K_v}(L \otimes_K K_v) \underset{\text{prop.}}{=} \sum_{w|v} \dim_{K_v}(L_w)$$

Compare to our ideal-theory fundamental identity:

$\mathfrak{p} \in \mathcal{O}_K$: Dedekind domain $\qquad \mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$

$L|K$: finite ext'n

and valuations $w_i = \frac{1}{e_i} v_{\mathfrak{P}_i}$ are normalized extensions of valuation $v = v_{\mathfrak{p}}$ on K.

We said before that e_i indeed agree with ramification indices $[w_i(L^\times) : v(K^\times)]$

since in discrete valuation $w_i(L^\times) = w_i(\mathfrak{P}) \cdot \mathbb{Z}$, $v(K^\times) = v(\mathfrak{p}) \cdot \mathbb{Z}$

What about f_i? Previously said $f_i = [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}]$

and when we pass to local fields then indeed these are isomorphic to $\mathcal{O}_w/\mathfrak{P}_w$ and $\mathcal{O}_v/\mathfrak{p}_v$.

Why does our factorization theorem before,

for all primes not dividing the conductor,

analyze $\mathfrak{p} \cdot \mathcal{O}_L$'s factorization by factoring min poly. f for α s.t. $L = K[\alpha]$.

mod p:

$$f(x) \equiv \bar{p}_1(x)^{e_1} \cdots \bar{p}_r(x)^{e_r} \pmod{p}$$

with $\deg(\bar{p}_i) =$ residual degree f_i, agree with new factorization theorem?