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Remember $\mathbb{F}_q((t))$: Laurent series in t (as opposed to $\mathbb{F}_q(t)$ - global function field - with rational functions in t.)

pf (sketch):

We construct continuous homoms.

$$g_n : \mathbb{Z}_p^{\oplus r} \longrightarrow \mathcal{U}^{(n)} = 1 + t^n \mathbb{F}_q[[t]]$$ for each n s.t. $\gcd(n,p)=1$

by choosing basis $\omega_1, \dots, \omega_r$ for $\mathbb{F}_q/\mathbb{F}_p$

$$(a_1, \dots, a_r) \longmapsto \prod_{i=1}^{r} (1+\omega_i t^n)^{a_i}$$

Then map $g := \prod_{(n,p)=1} g_n : \underbrace{\prod_{(n,p)=1} \mathbb{Z}_p^r}_{\cong \, \mathbb{Z}_p^{\mathbb{N}}, \text{ compact}} \longrightarrow \mathcal{U}^{(1)}$ $\quad g$: continuous.

show g is isomorphism.

Thm: If K is p-adic local field, $K^\times \simeq \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \underbrace{\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}_p^d}_{\mathcal{U}^{(1)}}$

if $\deg K/\mathbb{Q}_p = d$, $a \geq 0$.

pf: Key step is assertion that there exists continuous isomorphism

$$\log : \mathcal{U}^{(n)} \longrightarrow \mathfrak{p}_v^n = \pi^n \cdot \mathcal{O}_v \simeq \mathcal{O}_v$$

for n sufficiently large.

If we know this, then $\mathcal{O}_v$: int. closure of $\mathbb{Z}_p$ in K (proved ~~last~~ last time since K extends val. on $\mathbb{Q}_p$)

so has integral basis over $\mathbb{Z}_p$ of rank $\deg(K/\mathbb{Q}_p)$. (long ago proof, uses $\mathbb{Z}_p$ is P.I.D. proves $\underline{d} \cdot \mathcal{O}_v \subseteq$ integral basis where of size $\deg(K/\mathbb{Q}_p)$, $\underline{d}$ is disc.)

i.e. for n sufficiently large,

$\mathcal{U}^{(n)} \simeq \mathbb{Z}_p^d$. But $\mathcal{U}^{(1)}/\mathcal{U}^{(n)}$ finite, so done by structure of modules over P.I.D.

In the process, we've determined only remaining torsion elts are those in this $\mathcal{U}^{(1)}/\mathcal{U}^{(n)}$, i.e. have only p^{th} power roots of unity. $|\mathcal{U}^{(1)}/\mathcal{U}^{(n)}| = q^n = p^{rn}$

so $a \leq rn$

But not necessarily the whole gp.

Remains to prove key step: $\log : \mathcal{U}^{(n)} \xrightarrow{\sim} \mathfrak{p}_v^n$ for n suff. large. ($\mathfrak{p}_v^n \cong \mathcal{O}_v$)

Makes sense that we should seek out such a map since $\mathcal{U}^{(n)}$ structure as $\mathbb{Z}_p$ module is multiplicative, $\mathcal{O}_v$ structure is additive.

Define logarithm using power series. Funny issue: exponential is only defined (i.e. convergent) on sufficiently large powers of $\mathfrak{p}_v$

problem: valuations of factorials.

Proposition: For $\mathfrak{p}$-adic number field K, $\exists$ uniquely def'd hom.

$\log : K^\times \to K$ s.t.:

(1) $\log p = 0$

(2) If $1+x \in 1+\mathfrak{p}$, then

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Pf: First show (2) defines convergent power series, i.e. in this non-arch. setting, just have to check that $\left|\frac{x^n}{n}\right|_K \to 0$ as $n \to \infty$.

Since K is $\mathfrak{p}$-adic, it is an extension of $\mathbb{Q}_p$ and so, given according to the formula for extending $|\cdot|_p$

~~[illegible]~~

in terms of additive valuations: $v_K(\alpha) = \frac{1}{n} v_p(N_{K/\mathbb{Q}_p}(\alpha))$

$$v_K\left(\frac{x^n}{n}\right) = n \cdot v_K(x) - v_K(n)$$

Now: $v_K(x) > 0$ since $x \in \mathfrak{p}_{v_K}$

$v_K(n) = v_p(n) \leq \dfrac{\ln n}{\ln p}$

$\ln$: usual natural log.

($=$ if n is prime power.)

$$\text{so} \quad \geq \underbrace{\underbrace{n \cdot \frac{\ln c}{\ln p}}_{\substack{\text{creative way} \\ \text{to write } v_K(x) > 0 \\ \text{for some } c > 1}} - \frac{\ln n}{\ln p}}_{\frac{\ln(c^n/n)}{\ln p}} \to \infty \quad \text{as } n \to \infty.$$

Moreover, the identity of formal power series

$$\log((1+x)(1+y)) = \log(1+x) + \log(1+y),$$

converging if $1+x, 1+y \in \mathcal{U}^{(1)}$

gives us a homomorphism.

To define $\log$ on $K^\times$, note any elt. $\alpha \in K^\times$ has form

$$(*) \qquad \alpha = \pi^{v_\mathfrak{p}(\alpha)} \, w(\alpha) \cdot \langle \alpha \rangle$$

with $w(\alpha) \in \mu_{q-1}$

$\langle \alpha \rangle$: image in $\mathcal{U}^{(1)}$ elt. of $1 + \mathfrak{p}$

and $(\pi) = \mathfrak{p}$.

In particular $p = \pi^e \, w(p) \cdot \langle p \rangle$, so if we want $\log(p) = 0$,

then set $\log \pi = -\frac{1}{e} \underbrace{\log \langle p \rangle}_{}$ defined already by power series.

Now in $(*)$, $\log \alpha = v_\mathfrak{p}(\alpha) \log \pi + \log \langle \alpha \rangle$.

check that resulting definition gives continuous homom. $\log : K^\times \to K$.

$$\log(\alpha\beta) = v_{\mathfrak{p}}(\alpha\beta)\log\pi + \log\langle\alpha\cdot\beta\rangle$$

$$= (v_{\mathfrak{p}}(\alpha) + v_{\mathfrak{p}}(\beta))\log\pi + \log\langle\alpha\beta\rangle$$

Done if we can show $\langle\alpha\beta\rangle = \langle\alpha\rangle\langle\beta\rangle$

If $\alpha = \pi^{v_{\mathfrak{p}}(\alpha)} \cdot \omega(\alpha)\langle\alpha\rangle$

$\beta = \pi^{v_{\mathfrak{p}}(\beta)}\omega(\beta)\langle\beta\rangle$

then $\alpha\beta = \pi^{v_{\mathfrak{p}}(\alpha)+v_{\mathfrak{p}}(\beta)} \underbrace{\omega(\alpha)\omega(\beta)}_{\in \mu_{q-1}} \underbrace{\langle\alpha\rangle\langle\beta\rangle}_{= \langle\alpha\beta\rangle}$ $\leftarrow$ by uniqueness of rep'n. so indeed get homom.

now continuity need only be checked at 1, which is easy.

Final claim that any such cont. homom. $\lambda : K^\times \to K$ restricting to log as power series on $\mathcal{U}^{(1)}$, $\lambda(p) = 0$, agrees with our extension (i.e. uniquely det'd)

Use identity again: $p = \pi^e \omega(p)\langle p\rangle$

to show if the λ, log agree on μ_{q-1}, then must agree on π, so match on all $\alpha \in K^\times$ according to unique decomp. as above.

But $\lambda(\xi) = \dfrac{\lambda(\xi^{q-1})}{q-1}$ since λ homom. to K

$\xi \in \mu_{q-1}$

$= \dfrac{\lambda(1)}{q-1}$ and $\lambda(1) = 0$ according to power series defn. //

Theorem : $K/\mathbb{Q}_p$: p-adic local field , $\mathcal{O}_\wp$: valuation ring
(DVR so all ideals of form $\wp^k$ some k where $\wp$: unique max'l ideal)

write $p \cdot \mathcal{O}_\wp = \wp^e$. $e \geq 0$.

Then letting $\exp(x) := \sum_{m=0}^{\infty} \frac{x^m}{m!}$,

the maps: $\wp^n \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathcal{U}^{(n)}$ are inverse isomorphisms / homeomorphisms if $n > \frac{e}{p-1}$.

pf : Want to find n s.t. $\log : \mathcal{U}^{(n)} \longrightarrow \wp^n$. i.e. $v_\wp(\log(1+z)) = v_\wp(z) = n$ if $z \in \wp^n$

$\log(1+z)$ is limit of partial sums $\sum_{m=1}^{k} (-1)^{m+1} \frac{z^m}{m}$

whose valuation $\geq$ min of vals. of summands $\frac{z^m}{m}$.

Can use v_p for $\mathbb{Q}_p$ or $v_\wp$ for K, know they are related by, at least on $\mathbb{Q}_p$, $v_\wp = e \cdot v_p$ since $p \cdot \mathcal{O}_v = \wp^e$. So we can use $v_p := \frac{1}{e} v_\wp$ on K

$$v_p\left(\frac{z^m}{m}\right) - v_p(z) = (m-1)\, v_p(z) - v_p(m) \geq 0 \quad \text{so that}$$

WANT for n big enough $= v_\wp(z) = e \cdot v_p(z)$; $v_\wp(\log(1+z)) = v_\wp(z)$

Need an estimate on $\frac{v_p(m)}{m-1}$.

Write $m = p^a \cdot m_0$ with $\gcd(m_0, p) = 1$

$$\frac{v_p(m)}{m-1} = \frac{a}{p^a \cdot m_0 - 1} \leq \frac{a}{p^a - 1} = \frac{1}{p-1} \cdot \frac{a}{\underbrace{p^{a-1} + \cdots + p + 1}_{a \text{ factors}}} \leq \frac{1}{p-1}$$

So if we choose $v_p(z) \geq \frac{1}{p-1}$ i.e. $v_{\wp}(z) \geq \frac{e}{p-1}$, then difference of valuations is non-negative for all summands, so $v_{\wp}(\log(1+z)) = v_{\wp}(z) = n$ as desired.

In other direction, show if $x \in \wp^n$, then $\exp(x)$ converges provided that $n > \frac{e}{p-1}$. Key: Estimate valuations of $m!$ and can play similar games with summands to show, if $m > 1$,

$$v_p\left(\frac{x^m}{m!}\right) - v_p(x) \geq 0.$$

Then with such $n \gg 0$, $\exp \cdot \log(1+z) = 1+z$, $\log \exp(x) = x$ as identities of formal power series, all of which converge for this n. //