Use this in next theorem: Let $K = \hat{K}$ w.r.t valuation $|\cdot|_v$. Then $|\cdot|_v$ may be extended in a unique way to valuation of any algebraic extension $L|K$. The new absolute value on L is given by

$$|\alpha|_L = \sqrt[n]{|N_{L/K}(\alpha)|_v} \quad \text{if } \deg(L/K) \text{ finite, and}$$

$L = \hat{L}$ with respect to $|\cdot|_L$.

pf: Immediately reduce to non-arch. case, since only non-trivial ext'n of arch. abs. value on field is $K = \mathbb{R}$, $L = \mathbb{C}$ as $\mathbb{C}$ is algebraically closed.

Then true that $|\alpha|_{\mathbb{C}} = \sqrt[2]{|N_{\mathbb{C}/\mathbb{R}}(\alpha)|_{\mathbb{R}}}$. since $N_{\mathbb{C}/\mathbb{R}}(z) = z \cdot \bar{z}$.
(product over embeddings in sep. ext'n)

In non-arch. case, with $\deg(L/K)$ finite, $= n$.

Existence: check that given formula defines valuation on L, and restricts to initial valuation on K. Definition of $N_{L/K}$ makes many of these easy:

$\alpha = 0 \iff |\alpha|_L = 0$, $N_{L/K}$ multiplicative so $|\alpha\beta|_L = |\alpha|_L |\beta|_L$

and restricts to valuation on K since, if $\alpha \in K$, $N_{L/K}(\alpha) = \alpha^n$.

Remains to check (strong) triangle inequality:

$$|\alpha+\beta|_L \leq \max\{|\alpha|_L, |\beta|_L\}$$

divide by α or β to get equivalently:

show: if $|\alpha|_L \leq 1$ then $|\alpha+1|_L \leq 1$.

i.e. if $\sqrt[n]{|N_{L/K}(\alpha)|_v} \leq 1$ then $\sqrt[n]{|N_{L/K}(\alpha+1)|_v} \leq 1$. so remove n^{th} roots.

thus we must show if $N_{L/K}(\alpha) \in \mathcal{O}_v$: valuation ring of v,

then $N_{L/K}(\alpha+1) \in \mathcal{O}_v$.

claim : $\{ \alpha \in L \mid N_{L/K}(\alpha) \in \mathcal{O}_v \}$ = integral closure of $\mathcal{O}_v$ in L, $\overline{\mathcal{O}_v}$

(so it is a ring, and hence $\alpha \in \overline{\mathcal{O}}_v \Rightarrow \alpha+1 \in \overline{\mathcal{O}}_v$)

" $\supseteq$ " : if $\alpha \in \overline{\mathcal{O}}_v$, proved long ago that $N_{L/K}(\alpha) \in \mathcal{O}_v$.

" $\subseteq$ " : if $\alpha \in L^\times$, $N_{L/K}(\alpha) \in \mathcal{O}_v$, then write min. poly. for α /K

$f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_0$. $N_{L/K}(\alpha) = \pm a_0^m$ some $m \leq n$.

But if $\pm a_0^m \in \mathcal{O}_v \Rightarrow |\pm a_0^m|_v \leq 1$

$\Rightarrow |a_0|_v \leq 1 \Rightarrow a_0 \in \mathcal{O}_v$.

Applying our corollary to Hensel's lemma, then $f(x) \in \mathcal{O}_v[x]$, i.e. $\alpha \in \overline{\mathcal{O}}_v$.

Uniqueness : if $|\cdot|_L'$ is another valuation extending $|\cdot|_v$ with valuation ring $\mathcal{O}'$

(the claim also shows $\overline{\mathcal{O}}_v$ is valuation ring of $|\cdot|_L$.)

Show $\overline{\mathcal{O}}_v \subseteq \mathcal{O}'$ (and that this determines their equivalence)

if $\alpha \in \overline{\mathcal{O}}_v \setminus \mathcal{O}'$, then letting $f(x)$ as above be min. poly for α /K

then coeffs a_i of f are in fact in $\mathcal{O}_v$. $\alpha \notin \mathcal{O}'$ implies $\alpha^{-1} \in \mathfrak{p}'$ maximal ideal of $\mathcal{O}'$

But then can invert min poly:

$$1 = -a_1\alpha^{-1} - \dots - a_d(\alpha^{-1})^d \in \mathfrak{p}' \quad \text{↯}.$$

so we have $\overline{\mathcal{O}}_v \subseteq \mathcal{O}'$, i.e. $|\alpha|_L \leq 1 \Rightarrow |\alpha|_L' \leq 1$ which implies valuations are equivalent (and in fact, therefore equal since they agree on K)

Last statement of theorem: Resulting L, $|\cdot|_L$ is complete w.r.t.

This follows from Lemma: $K = \hat{K}$ with respect to $|\cdot|_v$ and let

V: fin. dim'l v.s.$/K$, dimension n, norm $\|\cdot\|$ ⇜ same properties as valuation – scalar mult.

$\|\alpha \underline{x}\| = |\alpha|_v \|\underline{x}\|$ is only dependence on $|\cdot|_v$.

Then maximum norm for any basis $v_1, \dots, v_n$ of V/K:

$$\| x_1 v_1 + \cdots + x_n v_n \|_{max} = \max \{ |x_1|_v, \dots, |x_n|_v \}$$

is equivalent to $\|\cdot\|$ on V and so

$$K^n \to V$$
$$(x_1, \dots, x_n) \mapsto x_1 v_1 + \cdots + x_n v_n$$

is a homeomorphism.

(i.e. completeness of V with respect to norm follows from completeness of K) w.r.t. $|\cdot|_v$

Now define what a "local field" is:

$$\left\{ \begin{array}{l} K: \text{field} = \hat{K} \text{ w.r.t.} \\ v: \text{archimedean, or} \\ \quad \text{non-arch. discrete} \\ \text{AND } \mathcal{O}_v/\mathfrak{p}_v \text{ finite} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{locally compact} \\ \text{fields } K \text{ w.r.t.} \\ \text{non-discrete topology} \end{array} \right\}$$

Recall X: top. space, assumed Hausdorff, is "locally compact" if every point $x \in X$ has a compact neighborhood.

(Require "non-discrete" since every field K w.r.t. discrete topology is locally compact trivially, don't just want all fields)

"$\mapsto$" Given $\hat{K}$ w.r.t. v, show $\hat{K}$ is locally compact w.r.t. topology induced by $|\cdot|_v$. If v archimedean, know $\mathbb{R}, \mathbb{C}$ locally compact

<u>Proposition</u>: $K = \hat{K}$ w.r.t. discrete non-arch val. v, then K is locally compact, and $\mathcal{O}_v$ compact.

<u>pf</u>: Need to show that $\mathcal{O}_v$ compact. Then 0 has compact nbhd, so we're done (as $a + \mathcal{O}_v$ is likewise compact nbhd of any $a \in K$)

Key <u>fact</u>: $\mathcal{O}_v \simeq \varprojlim \mathcal{O}_v/\mathfrak{p}_v^n$ both algebraically and topologically.

(discussed this for $\mathbb{Z}_p$ algebraically. topologically: $\mathcal{O}_v$ has metric $|\cdot|_v$ for $\varprojlim \mathcal{O}_v/\mathfrak{p}_v^n$, then each of gps $\mathcal{O}_v/\mathfrak{p}_v^n$ get discrete topology and then $\varprojlim \mathcal{O}_v/\mathfrak{p}_v^n$ is closed subset of $\prod_{n=1}^{\infty} \mathcal{O}_v/\mathfrak{p}_v^n$: given product topology with each constituent discrete.

so has induced topology. Much coarser than taking topology to consist of open sets $\prod_n U_n$, U_n open in $\mathcal{O}_v/\mathfrak{p}_v^n$.)

If we can show the key fact, then remember that $\mathcal{O}_v$ is discrete valuation ring if v discrete, so since we assume $\mathcal{O}_v/\mathfrak{p}_v$ finite and $\mathfrak{p}_v^n/\mathfrak{p}_v^{n+1} \simeq \mathcal{O}_v/\mathfrak{p}_v$ $\forall n$, then $\mathcal{O}_v/\mathfrak{p}_v^n$ finite. So compact in discrete topology.

Thus we recognize $\varprojlim \mathcal{O}_v/\mathfrak{p}_v^n$ as closed subset of compact, Hausdorff product so is compact, Hausdorff.

Key fact is proved by showing $\mathcal{O}_v \longrightarrow \varprojlim_n \mathcal{O}_v/\mathfrak{p}_v^n$ is isomorphism of gps and homeomorphism of top. spaces

$$\phi: \alpha \longmapsto \prod_{n=1}^{\infty} pr_n(\alpha)$$

pr_n canonical proj. mod $\mathfrak{p}_v^n$.

injective since $\ker(\phi) = \bigcap_n \mathfrak{p}^n = (0)$.

surjective since any elt. of target expressible as sequence of partial sums S_n with π-ary expansion.

Take $x = \lim_{n\to\infty} S_n \in \mathcal{O}_v$

now we need to show ϕ takes base of open nbhds of 0 in $\mathcal{O}_v$ to base in $\varprojlim_n \mathcal{O}_v/\mathfrak{p}_v^n$.

$\{\mathfrak{p}_v^n\}_{n=1}^{\infty}$ are base for 0 in $\mathcal{O}_v$

and $\phi(\mathfrak{p}_v^n) = \prod_{m>n} \mathcal{O}_v/\mathfrak{p}_v^m$, a base for 0 in $\prod_{n=1}^{\infty} \mathcal{O}_v/\mathfrak{p}_v^n$.

This completes "$\longrightarrow$" in claimed 1-1 correspondence.

"$\longleftarrow$" Given (non-discrete) locally compact field K, how to assoc. valuation?

K is locally compact abelian gp., so has unique up to constant translation invariant measure (Haar measure μ). Given automorphism ϕ of G

then $\mu \circ \phi$ and μ must agree up to constant $\in \mathbb{R}^\times_+$ (and importantly, that constant is indep. of choice of Haar measure)

If we take our automorphism to be multiplication in $K^\times$

$a \in K^\times : x \longmapsto ax$ for any fixed a,

define $\mathrm{mod}_K(a) = \dfrac{\mu(aX)}{\mu(X)}$ X: measurable set. Show $\mathrm{mod}_K(\cdot)$ is absolute value $K^\times \to \mathbb{R}^\times_+$. extend to K by $\mathrm{mod}_K(0) = 0$.

Weil shows either $\exists$ prime $p \in \mathbb{Z}$ s.t.

$\mathrm{mod}_K(p \cdot 1_K) < 1$ "p-field" $\rightsquigarrow$ image of mod_K discrete

or $\underset{K}{\underbrace{\text{it}}}$ is an algebra over $\mathbb{R}$ "$\mathbb{R}$-field" $\rightsquigarrow$ $\mathbb{R}$ or $\mathbb{C}$.

$\rightarrow$ See discussion on p.12 of Weil's Basic Number Theory

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Using characterization of non-arch. local fields as complete w.r.t. discrete valuation, finite residue field, classify local fields (Weil does this from other definition as locally compact field)

Theorem: Local fields are finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

Proof: first show all such fields are local fields: ✓ (extensions of either are complete w.r.t. unique extn from $\mathbb{Q}_p$ $\underset{K}{\underbrace{\text{or } \mathbb{F}_p((t))}}$ given by

$|\cdot|_L = \sqrt[n]{|N_{L/K}(\cdot)|_K}$ with $|\cdot|_K$ non-arch., discrete so $|\cdot|_L$ non-arch, discrete.)