Next, classify valuations of $\mathbb{Q}$ up to equivalence.

mult valuation $V: K^* \to \mathbb{R}_{\geq 0}$ with ~~[illegible]~~

(abs. value) $|\cdot|_*$

(i) $|a|_* = 0 \iff a = 0$

(ii) $|ab|_* = |a|_* |b|_*$

(iii) $|a+b|_* \leq |a|_* + |b|_*$

And such absolute value gives metric $d(x,y) = |x-y|$ on K, with induced topology.

Definition: Two valuations (abs. values) are equivalent if they induce the same topology.

Proposition: Two valuations $|\cdot|_1$ and $|\cdot|_2$ induce same topology $\iff$ $\exists$ real # $s > 0$ with $|x|_1 = |x|_2^s \quad \forall x \in K$

Pf: $(\Leftarrow)$ if $|\cdot|_1 = |\cdot|_2^s$ with $s > 0$, then they must define same topology. (open sets about 0 are $x \in K$ s.t. $|x| < c$, $c \in \mathbb{R}_{>0}$.)

$(\Rightarrow)$ if $|x| < 1$, then $\{x^n\}$ converges in topology induced from $|\cdot|$. But then if $|\cdot|_1$ and $|\cdot|_2$ are equivalent,

$$|x|_1 < 1 \Rightarrow |x|_2 < 1.$$

Given $y \in K$ with $|y|_1 > 1$ then given any $x \neq 0 \in K$ $\exists\, \alpha \in \mathbb{R}$ with $|x|_1 = |y|_1^\alpha$. Let $\{\frac{m_i}{n_i}\}_i \in \mathbb{Q}$ converging to α from above. Take $n_i > 0$.

Then $|x|_1 = |y|_1^{\alpha} < |y|_1^{m_i/n_i} \quad \forall i \Rightarrow \left| \frac{x^{n_i}}{y^{m_i}} \right|_1 < 1 \quad \forall i$

$\Rightarrow \left| \frac{x^{n_i}}{y^{m_i}} \right|_2 < 1$ (since $|\cdot|_1, |\cdot|_2$ equivalent) $\forall i$

$\Rightarrow |x|_2 \le |y|_2^{m_i/n_i}$ $\forall i$ $\Rightarrow |x|_2 \le |y|_2^{\alpha}$.

Now repeat with sequence converging to α from below. Get: $|x|_2 \geq |y|_2^{\alpha}$

so $|x|_2 = |y|_2^{\alpha}$. But then $\frac{\log|x|_1}{\log|x|_2} = \frac{\log|y|_1}{\log|y|_2} =: s$

and so $|x|_1 = |x_2|^s$. Since $|y|_1 > 1 \Rightarrow |y|_2 > 1$, then $s > 0$. //

Note in the course of the proof, we showed that $|\cdot|_1 \sim |\cdot|_2 \iff$

$$|x|_1 < 1 \Rightarrow |x|_2 < 1 \quad \forall x \in K.$$

Use this in:

Approximation Theorem: Let $|\cdot|_1, \dots, |\cdot|_n$ be pairwise inequivalent abs. values on K. $a_1, \dots, a_n \in K$, then for every $\epsilon > 0$, $\exists x \in K$ s.t.

$$|x - a_i|_i < \epsilon \quad \forall i = 1, \dots, n$$

(Neukirch: Version of Chinese Remainder Thm. $\epsilon = \frac{1}{p^k}$ for $x \equiv a_i \bmod p^k$ w.r.t. p-adic valuation)

pf: Show $\exists y \in K$ with $|y|_1 > 1$, $|y|_n \le 1$, and so by induction $\exists z \in K$ with $|z|_1 > 1$, $|z|_i < 1$ $i = 2, \dots, n$.

then $z^m / 1+z^m \longrightarrow 1$ w.r.t. $|\cdot|_1$. Pick m_0 large,

$\longrightarrow 0$ w.r.t. $|\cdot|_i$ $i=2,\dots,n$ call $z_1 := \dfrac{z^{m_0}}{1+z^{m_0}}$

then at last pick $x = a_1 z_1 + \cdots + a_n z_n$.

To show initial y exists, then use fact that $|\cdot|_1 \not\sim |\cdot|_n \Rightarrow$

$\exists$ α with $|\alpha|_1 < 1$, $|\alpha|_n \geq 1$ set $y = \beta/\alpha$. This is desired y.

β with $|\beta|_n < 1$, $|\beta|_1 \geq 1$

Classify valuations into two groups based on whether $\{|n|\}_{n \in \mathbb{Z}_{>0}}$ is

bounded or unbounded

"non-archimedean" "archimedean"

Easy fact: An absolute value is non-archimedean $\Longleftrightarrow$ $|x+y| \leq \max\{|x|,|y|\}$ $\forall$ $x,y \in K$.

$(\Leftarrow)$ $|n| = |\underbrace{1+\cdots+1}_{n \text{ times}}| \leq |1| = 1$

$(\Rightarrow)$ binomial theorem relating n to $|x+y|$: abs. bound on $|m|$ with $m = \binom{n}{i}$'s $\forall m$

$$|x+y|^n \leq \sum_i \left|\binom{n}{i}\right| |x|^i |y|^{n-i} \leq N \cdot (n+1) |x|^n$$

assume w.l.o.g. $|x| \geq |y|$ i.e. $|x|$ is max.

# of terms in sum.

Now take n^{th} roots and then lim as $n \to \infty$.

$$|x+y| \leq \underbrace{N^{1/n} (n+1)^{1/n}}_{\to 1 \text{ as } n \to \infty} |x|$$

Use this dichotomy in proof of today's main result:

<u>Thm</u>: Every valuation of $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some p or to $|\cdot|_\infty$, the usual Archimedean absolute value. (Assuming non-trivial)

<u>pf</u>: Suppose $|\cdot|$ is non-archimedean. Then $|n| = |\underbrace{1+\cdots+1}_{n \text{ times}}| \leq 1$ $(*)$

and $\exists$ prime p with $|p| < 1$, else any rational $p_1^{e_1}\cdots p_r^{e_r}$, $e_i \in \mathbb{Z}$ $(\neq 0)$

has abs. value 1. (trivial absolute value)

<u>claim</u>: $\mathfrak{a} := \{ a \in \mathbb{Z} \mid |a| < 1 \} = p \cdot \mathbb{Z}$ (with $|p|<1$ as above)

<u>pf</u>: $\mathfrak{a}$ is ideal in $\mathbb{Z}$ by definition, and $\mathfrak{a} \supseteq p\mathbb{Z}$ but $\mathfrak{a} \neq \mathbb{Z}$ and $p\mathbb{Z}$ maximal, so $\mathfrak{a} = p\mathbb{Z}$.

Given $a \in \mathbb{Z}$, write $a = p^m \cdot b$ with $\gcd(b,p) = 1$. Then $b \notin p\mathbb{Z}$ and $|b| \leq 1$ by $(*)$ so $|b| = 1$. $\Rightarrow$ $|a| = |p|^m$

$= |a|_p^s$

where $s = \dfrac{-\log |p|}{\log p}$

i.e. is equivalent to $|\cdot|_p$ on $\mathbb{Z}$ and hence on $\mathbb{Q}$.

Now suppose $|\cdot|$ is archimedean.

<u>claim</u>: $|m|^{1/\log m} = |n|^{1/\log n}$ for every pair m,n of positive integers (>1)

If we can show the claim, then let c be this constant $= |n|^{1/\log n}$ $\forall n$.

so that $|n| = c^{\log n}$ and taking s s.t. $c = e^s$ implies

for $x = \frac{a}{b}$, $|x| = e^{s \log x} = |x|_\infty^s$ / Now to prove claim...

Proof of claim that $|m|^{1/\log m} = |n|^{1/\log n} \quad \forall\, m,n \in \mathbb{Z}_{>1}$

$m = a_0 + a_1 n + \cdots + a_r n^r$ with $a_i \in \{0, \ldots, n-1\}$, $n^r \leq m$.

then since $r \leq \log m / \log n$,

$$|m| \leq \sum_i |a_i||n^i| \leq |n|^r \cdot \sum_i |a_i|$$

using that $n > 1$
$\Rightarrow |n| > 1$

~~get this because, for archim. valuation, any subsequence of ~~~~[illegible]~~~~ integers $\to \infty$ must have unbounded valuations.~~ (Not nec. true according to our defin....)

see Weil's Basic Number Theory for fix.

MISTAKE IN NEUKIRCH?