Fubini's theorem in $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} f(x,y)\ |d(x,y)| = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\ |dy|\ |dx|$$

if inner integrand $y \mapsto f(x,y)$ is integrable.

No different in $\mathbb{R}^{n+m}$: $\underline{x} \in \mathbb{R}^n$, $\underline{y} \in \mathbb{R}^m$

$$\int_{\mathbb{R}^{n+m}} f(\underline{x},\underline{y})\ \underbrace{|d^n\underline{x}|\ |d^m\underline{y}|}_{|d^{n+m}(\underline{x},\underline{y})|} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} f(\underline{x},\underline{y})\ |d^m\underline{y}|\ |d^n\underline{x}|$$

if $\underline{y} \mapsto f(\underline{x},\underline{y})$ integrable.

Plan: if given integral over $\mathbb{R}^3$, use Fubini's theorem twice to break into 3 one-dimensional integrals.

Example: Volume of sphere of radius r. (3 dim'l ball)

(Rectangular coordinates are not simplest way to solve this problem, so we'll have better coordinate systems for handling this later. But volumes in other coordinates come with complications)

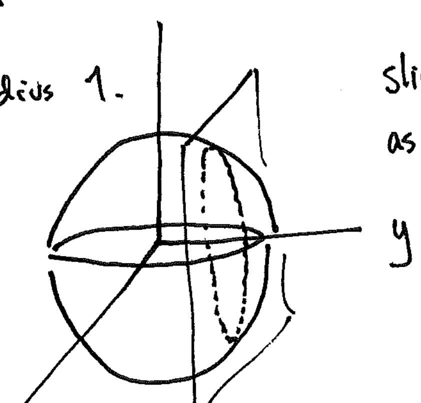
First, if a set $A \subseteq \mathbb{R}^n$ has volume and $r \in \mathbb{R}$, then rA is pavable and $\mathrm{vol}_n(rA) = |r|^n \cdot \mathrm{vol}_n(A)$.

(reason: true for cubes. and if $C_{k,N} \subset A$ then $r \cdot C_{k,N} \subset r \cdot A$)

By the way, what is the definition of the set rA ?

Now it suffices to find volume of ball of radius 1.

$$\int_{\mathbb{R}^3} 1_{B_1(0)}\ |d^3\underline{x}|$$



slice in y as outer integration

Then $y \in [-1,1]$.

equation of (unit) sphere: $x^2+y^2+z^2=1$ (i.e. all points distance 1 from origin in $\mathbb{R}^3$)

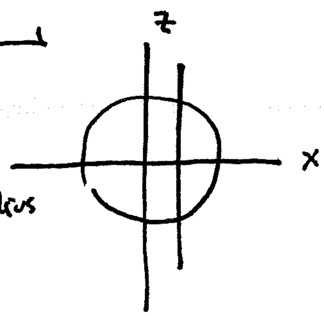
Fix $y=y_0$ get circle $x^2+z^2 = 1-y_0^2$.

i.e. circle of radius $\sqrt{1-y_0^2}$.

Fubini's thm: $$\int_{\mathbb{R}^3} = \int_{\mathbb{R}}\int_{\mathbb{R}^2} \underbrace{1_{B_1(0)}}_{1_{[-1,1]}(y)\cdot 1_{\text{circle in } x,z \text{ with rad. } \sqrt{1-y^2}}} |d(x,z)|\,|dy|$$

$$= \int_{-1}^{1}\int_{\substack{\text{circle}\\ \text{of radius}\\ \sqrt{1-y^2}}} dx\,dz\;\,dy$$

Go further: In $\mathbb{R}^2$ circle of radius r



$$= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dz\,dx.$$

$$= 4\int_0^r\int_0^{\sqrt{r^2-x^2}} dz\,dx$$

(trig substitution.)

or use scaling again:

Area (circle of radius $\sqrt{1-y^2}$)

$= (\sqrt{1-y^2})^2 \cdot \underbrace{\text{Area (circle of radius 1)}}_{\substack{\text{compute via easier integral}\\ \text{or take as known, } = \pi}}$.

Get $= \displaystyle\int_{-1}^{1}\pi(1-y^2)\,dy = \pi\left(y-\frac{1}{3}y^3\right)\Big|_{-1}^{1} = \pi\left(\frac{2}{3}--\frac{2}{3}\right)$

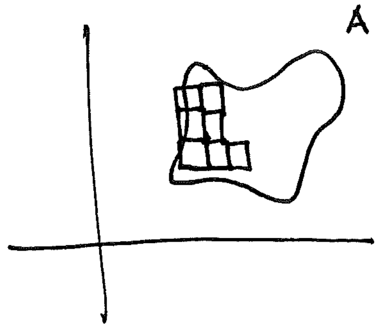
$= \frac{4}{3}\pi$. ✓

Do inductive pf + Fubini's thm to prove volume in

$\mathbb{R}^{2k}$: $\dfrac{\pi^k}{k!}$ $\qquad \mathbb{R}^{2k+1}$: $\pi^k\dfrac{k!\,2^{2k+1}}{(2k+1)!}$

Why is Fubini's theorem true? Crude answer - integrals are approximated by finite sets of cubes

E.g. in $\mathbb{R}^2$



Doesn't matter if we sum them up in any order.

Doing y integral first, like counting up boxes in fixed column.

x integral first, counting up boxes in rows.

Just need to make this precise.

f integrable so $U_N(f) \geq L_N(f)$ with equality as $N \to \infty$

Use squeeze theorem if we can put pieces of Fubini's theorem in between.

If $y \mapsto f(x,y)$ is inner integrand. Do $U_{N'}(y \mapsto f(x,y))$ then take upper sum of this. Claim:

$$U_N(f) \geq \underbrace{U_N}_{}\left(U_{N'}\left(y \mapsto f(x,y)\right)\right) \quad \text{for } N' \geq N.$$

etc.

need to write out definitions to prove this.

(taking maxes over restricted sets, so $\geq$ results)

Version of Fubini's thm. that results: $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\underline{x}, \underline{y} \mapsto f(\underline{x}, \underline{y})$ integrable

then
$$\int_{\mathbb{R}^n} U(\underline{y} \mapsto f(\underline{x},\underline{y}))\,|d^n \underline{x}| = \int_{\mathbb{R}^n} L(\underline{y} \mapsto f(\underline{x},\underline{y}))\,|d^n x|$$

if these are integrable, then replace w/ integrals

$$= \int_{\mathbb{R}^m} U(\underline{x} \mapsto f(\underline{x},\underline{y}))\,|d^m \underline{y}| = \int_{\mathbb{R}^m} L(\underline{x} \mapsto f(\underline{x},\underline{y}))\,|d^m \underline{y}| = \int_{\mathbb{R}^{n+m}} f\,|d^{n+m}(\underline{x},\underline{y})|$$