Orientation preserving parametrizations:

Integrate Work form $F\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{bmatrix}-yz\\xz\\2y+1\end{bmatrix}$ over the

helix $\gamma(t) = \begin{bmatrix}\cos t\\ \sin t\\ t\end{bmatrix}$ $\quad 0<t<2\pi$.

Find orientation for 1-manifold in $\mathbb{R}^3$. Idea 1: Pick vector in tangent space and use dot product.

(why does this only work for 1-manifolds? Imagine space of 2-manifold ...)

e.g. $\gamma'(t) = \begin{bmatrix}-\sin t\\ \cos t\\ 1\end{bmatrix}$

is tangent vector (non-zero) $\forall\, t \in (0, 2\pi)$.

So choosing this, $\quad \gamma'(t)\cdot v = |\gamma'(t)|^2 = 2. \neq 0.$ ✓.

write $v = \gamma'(t)$

also (preserves orientation, as always +)

Now compute: $\displaystyle\int_0^{2\pi} \underbrace{W_F(\underline{x})(\gamma'(t))}_{\text{defined as } F(\underline{x})\cdot\gamma'(\underline{t})}\,dt$, $\quad \underline{x} = \gamma(t)$

$$= \int_0^{2\pi} \begin{bmatrix}-t\sin t\\ t\cos t\\ 2\sin t+1\end{bmatrix}\cdot\begin{bmatrix}-\sin t\\ \cos t\\ 1\end{bmatrix} dt$$

$$= \int_0^{2\pi} \left[t(\sin^2 t + \cos^2 t) + 2\sin t + 1\right] dt$$

$$= \frac{4\pi^2}{2} - 2(\cos 2\pi - \cos 0) + 2\pi .$$

Other orientation:

Pick two vectors not in tangent space.

At each point, could pick v_1, v_2 s.t.

$\gamma'(t)\cdot(v - \gamma(t)) = 0.$

then try

$$\operatorname{sgn}\left(\det\begin{pmatrix} | & | & | \\ v_1 & v_2 & \gamma'(t) \\ | & | & | \end{pmatrix}\right)$$

e.g. $\gamma'(t) \cdot \gamma(t) = -\sin t \cdot \cos t + \cos t \cdot \sin t + 1 \cdot t$

$= t$

So need $\gamma'(t) \cdot v = t$ with $-\sin t \cdot v_1 + \cos t \cdot v_2 + v_3$

Choose: $v_1 = \cos t, \ v_2 = \sin t, \ v_3 = t$

$v_1' = 2 \cos t \quad v_2' = 2 \sin t, \quad v_3' = t$

(dot prod. is $2 + t^2$.)

Famous form: Force exerted on charged particle with charge q at position $\underline{x} = (x_1, x_2, x_3)$ and time t, with velocity $\underline{v}$ depends on magnetic and electric fields.

observed that $F(\underline{x}, t) = q \cdot \left(E(\underline{x}, t) + \frac{\underline{v}}{c} \times B(\underline{x}, t) \right)$

c = speed of light.

Problem:

F not a vector field

if charge is stationary ($\underline{v} = \underline{0}$) and $E = 0$, then

$F = 0$. (indep. of $\underline{x}, t$.)

choose another coord. system moving with constant speed with respect to first. Then still no force on particle, but $E \neq 0$. contradiction!

(if really coordinate invariant)

Use form fields instead. Faraday 2-form: $W_{\vec{E}} \wedge c\,dt + \Phi_{\vec{B}}$

Maxwell 2-form $W_{\vec{B}} \wedge c\,dt - \Phi_{\vec{E}}$

Maxwell's equations: Exterior deriv. on these is 0.

Given f with $x \mapsto |xf(x)|$ L-integrable,

show $\frac{d}{dt} \hat{f}(t) = \widehat{ixf}(t)$

where $\hat{f}(t) \overset{\text{def}}{=} \int_{\mathbb{R}} f(x) e^{ixt} \, dx$

<u>Ans</u>: differentiate under integral sign. When is it justified?

Dominated convergence theorem.

show diff. quotient bounded by some L-int. $g(x)$. whenever, $\forall$ s.t. $|h| < \delta$ (fixed δ)

$$\left| f(x) \frac{e^{ix(t+h)} - e^{ixt}}{h} \right| \le g(x)$$

choose $g = |xf(x)|$

<u>Changes of vars</u>: axis of symmetry $\rightsquigarrow$ cylindrical

point (or "center") of symmetry $\rightsquigarrow$ spherical

slices of object in smaller dimension.

e.g. compute volume of $\left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1+z}\right)^2 \le 1 \right\}$

with $z \in (-1,1)$

slices are ellipses for fixed z. ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ param. by $\begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} a r \cos\theta \\ b r \sin\theta \end{pmatrix}$