Semester as progression to Stokes' theorem: $X \subseteq M$ compact, good boundary

then $\displaystyle\int_{\partial X} \varphi = \int_X d\varphi$, φ : class C^2-form on X ($k-1$ if $\dim(M) = k$)

① Integration using Riemann sums

(dyadic pavings of compactly supp. functions with compact values.

3 key results: ⓐ integrability (iff discontinuous on a set of measure 0.)

ⓑ Fubini's thm (iterated integration)

ⓒ Change of vars. theorem

lots of hypotheses, on change of vars function Φ

(injective on $X - \partial X$,

C^1 with Lipschitz deriv.

$D\Phi$ invertible at all $x \in X - \partial X$.)

½ step to Lebesgue integrability

Infinite sum of Riemann integrable functions.

4 ~~3~~ key results: ⓐ well definedness

ⓑ dominated convergence theorem

ⓒ/ⓓ Fubini's thm, change of vars.

So now can integrate more general functions (not nec. compact support, compact values)

L-int. functions

$f_k \to f$, and $\exists$ L-int. F s.t. $|f_k(x)| \le F(x)$ a.e.

then f is integrable and equal to $\displaystyle\lim_k \int f_k$.

(leads to diff. under integral sign.)

(2) integration on manifolds.

two key topics: (a) relaxed parametrizations

(b) curvature.

Gauss' theorem on area of disc on surface.

(error given by curvature)

$\text{Area}(D_r(p)) = \pi r^2 - \frac{\pi K(p)}{12} r^4 + o(r^4)$

Make this our pf.

(Each chapter has important "independence of param" using change of vars but won't pick it each time)

$U \subseteq \mathbb{R}^n$ with "bad place X"

$\gamma(U) \supset M \supset \gamma(\underset{\sim}{U - X})$

like before: one-one

C^1 w/ Lipschitz deriv. (loc.)

$D\gamma$ one-one $\forall\ u \in U - X$

k-volume of X

and $\gamma(X) \cap$ compact

$= 0 \quad \forall$ compact.

Lipschitz deriv if function is C^2 (even had way of producing Lipschitz ratio from second partials long time ago)

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(3) Oriented integration on manifolds. (~40% of test)

- differential forms (and form fields) (evaluate them, take exterior deriv, simple properties)
- orientations (definition + key examples (use form to make orientation), consistent with param:

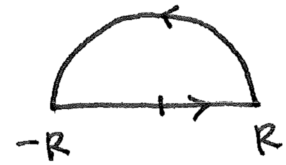
$\Omega(D\gamma(u)) = 1 \quad \forall\ u \in U - X$)

$\gamma : U \to M$.

- integrating k-forms over parametrized domain

- Stokes' theorem (pf. is one for domain nice w.r.t. coordinate axes) Prop. 6.9.7.

- "Favorite forms"

Example: $\displaystyle\int_C xy\,dy + x^3\,dx$ where $C =$ 

Semi-circular curve with radius R.

(std. counterclockwise orient.)

Parametrize each piece. Or use Stokes' theorem in $\mathbb{R}^2$.

Take $d(f\,dx + g\,dy)$ to make 2-form. In general, only

$$d_y f \; dy \wedge dx + d_x g \; dx \wedge dy \quad \text{survive}$$

$$= (D_x g - D_y f)\; dx \wedge dy.$$

In $\mathbb{R}^2$, identity:

$$\int_C f\,dx + g\,dy = \int_S (D_x g - D_y f)\; \underbrace{dx \wedge dy}$$

(with $\partial S = C$)

is called "Green's theorem"

in $\mathbb{R}^2$, if we have open set, can use x, y as coords. on manifold.

$$dx \wedge dy = |d^2(x,y)|$$

Green's / Stokes' thm $\Rightarrow$

$$\int_C xy\,dy + x^3\,dx = \int_{S\,:\,\text{semi-circle}} y\,dx \wedge dy.$$

$$\underset{\text{POLAR!}}{=} \int_0^{\pi}\int_0^R r\sin\theta \; r\,dr\,d\theta$$

$$= \frac{R^3}{3} \cdot [-\cos\theta]\Big|_0^{\pi}$$

$$= \boxed{2R^3/3}$$

A little bit fast about checking that param. using polar coords was consistent with orientation.

$\Omega(|D\gamma(u)|) = 1 \quad \forall\, u \in (0,\pi) \times (0,R).$

One more example: Compute the flux through the unit cube of the vector field $\Phi(x,y,z) = \begin{bmatrix} xy^2 + z \\ 4yz - xz \\ x\cos y \end{bmatrix}$ with standard orientation on cube. (outward pointing normal)

Again, Stokes' theorem:

(nice to remember that flux form

$d: \underbrace{\Phi(x,y,z)}_{\substack{\text{2-form we get} \\ \text{from } \det\left[F_\Phi \; v \; w\right]}} \longmapsto$ Mass form $M_{\underbrace{\mathrm{div}(\Phi)}_{\nabla\cdot\Phi}}$)

F_Φ: vector field assoc. to Φ

then

$$\int_{\partial C} \Phi = \int_C d\Phi$$

"divergence theorem"

$$= \int_C M_{\mathrm{div}(\Phi)}$$

$$= \int_C (y^2 + 4z)\, dx\, dy\, dz$$

$$= \int_0^1 \frac{1}{3}y^3 + 4yz \Big|_0^1 \, dz$$

$$= \int_0^1 \frac{1}{3} + 4z \; dz$$

$$= 1/3 + 2 = \boxed{7/3}$$

claim: $\Phi_F(\underline{x})(v,w)$

$$= F(\underline{x}) \cdot \underbrace{N(\underline{x})}_{\text{unit normal vector}} \, |d^2\underline{x}|$$

($N(\underline{x})\, |d^2\underline{x}|\, (\underline{v},\underline{w})$

$$= N(\underline{x}) \cdot |\underline{v} \times \underline{w}|$$

$$= \underline{v} \times \underline{w} \;)$$

so just check that $F(\underline{x}) \cdot (v \times w) = \det(F \; v \; w)$.